Game Theory

Chapter 4
N-person Nonzero Sum Games with A Continuum Of Strategies

Instructor: Chih-Wen Chang

Contents

4.1 The Basic

The Formal Definition of Nash Equilibrium

- There are *N* players in a game, we assume that each player has her or his own payoff function depending on her or his choices of the other players.

- Suppose that the strategies must take values in sets $Q_i, i = 1, \ldots, N$ and the payoffs are real-valued functions

$$u_i : Q_1 \times \cdots \times Q_N \to \mathbb{R}, \qquad i = 1, 2, \ldots, N.$$

Definition 4.1.1

- **Definition 4.1.1**
  - *A collection of strategies* $q^* = (q_1^*, \ldots, q_n^*)$ *is a Nash equilibrium for the game with payoff functions* $\{u_i(q)\}, i = 1, \ldots, N$ *, if for each player* $i = 1, \ldots, N$*, we have*

$$u_i(q_1^*, \ldots, q_{i-1}^*, q_i^*, q_{i+1}^*, \ldots, q_N^*) \geq u_i(q_1^*, \ldots, q_{i-1}^*, q_i, q_{i+1}^*, \ldots, q_N^*), \textit{ for all } q_i \in Q_i.$$

  - A Nash equilibrium consists of strategies that are all best response to each other.
  - No player can do better by deviating from a Nash point, assuming that no one else deviate.
    - ***It doesn't mean that group of players couldn't do better by playing something else.***

A Saddle Point of Two-person Zero Sum Game

- **Remark**
  1. If there there are two players and $u_1 = -u_2$, then a point (q_1^*, q_2^*) is a Nash point if

$$u_1(q_1^*, q_2^*) \geq u_1(q_1, q_2^*) \text{ and } u_2(q_1^*, q_2^*) \geq u_2(q_1^*, q_2), \forall (q_1, q_2).$$

But then

$$-u_1(q_1^*, q_2^*) \geq -u_1(q_1^*, q_2) \Longrightarrow u_1(q_1^*, q_2^*) \leq u_1(q_1^*, q_2),$$

and putting these together we see that

$$u_1(q_1, q_2^*) \leq u_1(q_1^*, q_2^*) \leq u_1(q_1^*, q_2), \forall (q_1, q_2).$$

This says that (q_1^*, q_2^*) is a saddle point of the two-person zero sum game.

Find Nash Point by Calculus

2. The steps involved in determining $(q_1^*, q_2^*, \ldots, q_n^*)$ as a Nash equilibrium are the following:
   - (a) Solve $$\frac{\partial u_i(q_1, \ldots, q_n)}{\partial q_i} = 0, \quad i = 1, 2, \ldots, n.$$
   - (b) Verify that q_i^* is the only stationary point of the function
   - (c) Verify $q \mapsto u_i(q_1^*, \ldots, q_{i-1}^*, q, q_{i+1}^*, \ldots, q_n^*)$ for $q \in Q_i$.

   evaluated at $$\frac{\partial^2 u_i(q_1, \ldots, q_n)}{\partial q_i^2} < 0, \quad i = 1, 2, \ldots, n,$$

(a), (b), and (c) hold for $(q_1^*, q_2^*, \ldots, q_n^*)$ $\overset{\Longrightarrow}{\not\Longleftarrow}$ Nash equilibrium $q_1^*, \ldots, q_n^*$.

The above are sufficient but the necessary conditions !!

Find Nash Point by Calculus (cont'd)

3. For the calculus approach we take the partial of u_i with respect to q_i, not the partial of each payoff function with respect to all variable.

   We are not trying to maximize each payoff function over all the variables, but each payoff function to each player as a function only of variable they control, namely, q_i.

EXAMPLE 4.1

- Two persons game with pure strategy sets $Q_1 = Q_2 = \mathbb{R}$ and payoff function

$$u_1(q_1, q_2) = -q_1 q_2 - q_1^2 + q_1 + q_2 \text{ and } u_2(q_1, q_2) = -3q_2^2 - 3q_1 + 7q_2.$$

Then

$$\frac{\partial u_1}{\partial q_1} = -q_2 - 2q_1 + 1 \text{ and } \frac{\partial u_2}{\partial q_2} = -6q_2 + 7.$$

There is one and only one solution of these, and it is given by $q_1 = -\frac{1}{12}, q_2 = \frac{7}{6}$. Finally, we have

$$\frac{\partial^2 u_1}{\partial q_1^2} = -2 < 0 \text{ and } \frac{\partial^2 u_2}{\partial q_2^2} = -6 < 0,$$

and so $(q_1, q_2) = (-\frac{1}{12}, \frac{7}{6})$ is indeed a Nash equilibrium.

EXAMPLE 4.2

- Do politicians pick a position on issues to maximize their votes? Suppose that voter preferences on the issue are distributed from [0,1] according to a continuous probability density function $f(x) > 0$ and $\int_0^1 f(x)\ dx = 1$. The density $f(x)$ approximately represents the percentage of voters who have preference $x \in [0,1]$ over the issue.

- The question a politician might ask is: "Given f, what position in [0,1] should I take in order to maximize the votes that I get in an election against my opponent?" The opponent also asks the same question.

EXAMPLE 4.2 (cont'd)

- **We assume that voters will always vote for the candidate nearest to their own positions.**

- Let's call the two candidates I and II, and let's take the position of player I to be $q_1 \in [0, 1]$ and for player II, $q_2 \in [0, 1]$. Let *V* be the random variable that is the position of a randomly chosen voter so that *V* has continuous density function f.

EXAMPLE 4.2 (cont'd)

- The payoff functions for player I and II is given by

$$u_1(q_1, q_2) \equiv \begin{cases} Prob(V \leq \frac{q_1 + q_2}{2}) & \text{if } q_1 < q_2; \\ \frac{1}{2} & \text{if } q_1 = q_2; \\ Prob(V > \frac{q_1 + q_2}{2}) & \text{if } q_1 > q_2. \end{cases}$$

$$u_2(q_1, q_2) \equiv 1 - u_1(q_1, q_2).$$

  - This is a problem with a discontinuous payoff pair, and we cannot simply take derivatives and set them to zero to find the equilibrium.
  - Let $\gamma \equiv \frac{q_1 + q_2}{2}.$

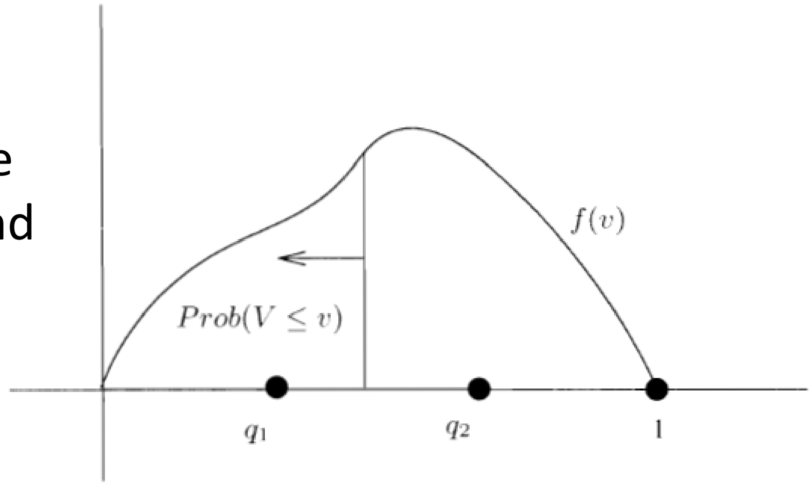


Figure 4.1 Area to the left of v is candidate I's percentage of the vote.

EXAMPLE 4.2 (cont'd)

- It seems that player I should not go further to the right than q_2, but should equal q_2. In addition, we should have

$$Prob(V \le \gamma) = Prob(V \ge \gamma) = 1 - Prob(V \le \gamma) \implies Prob(V \le \gamma) = \frac{1}{2}.$$

, namely

$$F_V(\gamma) \equiv P(V \le \gamma) = \int_0^\gamma f(x)\, dx = \frac{1}{2}.$$

,then we get $\gamma = \gamma^*$, which is the **median** of the random variable *V.*

- If γ^* is the median of the voter positions then (γ^*, γ^*) is a Nash equilibrium for the candidates, $u_i(\gamma^*, \gamma^*) = \frac{1}{2}$.

EXAMPLE 4.2 (cont'd)

- How do we check this? We need to verify directly from the definition of equilibrium that

$$u_1(\gamma^*, \gamma^*) = \frac{1}{2} \geq u_1(q_1, \gamma^*) \text{ and } u_2(\gamma^*, \gamma^*) = \frac{1}{2} \geq u_2(\gamma^*, q_2) \quad , \quad q_1, q_2 \in [0, 1]$$

  - If we assume $q_1 > \gamma^*$, then

$$u_1(q_1, \gamma^*) = P\left(V \geq \frac{q_1 + \gamma^*}{2}\right) \leq P\left(V \geq \frac{\gamma^* + \gamma^*}{2}\right) = P(V \geq \gamma^*) = \frac{1}{2}.$$

  - If, on the other hand, $q_1 < \gamma^*$, then

$$u_1(q_1, \gamma^*) = P\left(V \leq \frac{q_1 + \gamma^*}{2}\right) \leq P\left(V \geq \frac{\gamma^* + \gamma^*}{2}\right) = P(V \geq \gamma^*) = \frac{1}{2}.$$

    and we are done.

Theorem 4.1.2

- **Theorem 4.1.2**
  - *Let $Q_1 \subset \mathbb{R}^n$ and $Q_2 \subset \mathbb{R}^m$ be compact and convex sets. Suppose that the payoff functions $u_i : Q_1 \times Q_2 \to \mathbb{R}^n \times \mathbb{R}^m,\ i = 1, 2$, satisfy*
    - u_1 *and* u_2 *are continuous.*
    - $q_1 \mapsto u_1(q_1, q_2)$ *is concave for each fixed* q_2 *.*
    - $q_2 \mapsto u_2(q_1, q_2)$ *is concave for each fixed* q_1 *.*

    Then, there is a Nash equilibrium for (u_1, u_2) *.*

- Von Neumann's theorem says roughly that a function f(x,y) that is concave in x and convex in y will have a saddle point.
  - The connection with the Nash theorem is made by noticing that f(x,y) is the payoff for player I and –f(x,y) is the payoff for player II.

Theorem 4.1.2 (Cont'd)

- So, if $y \mapsto f(x, y)$ is convex , then $y \mapsto -f(x, y)$ is concave.
- ***Nash's result is a true generalization of the von Neumann minimax theorem.***
- ***Also, the Nash theorem is only a sufficient condition, not a necessary condition for a Nash equilibrium.***

4.2 Economics Applications of Nash equilibrian Problems

Cournot Duopoly

- Cournot developed one of the earliest economic models of the competition between two firms. Suppose that there are two companies producing the same gadget. Firm 2 = 1,2 chooses to produce the quantity $q_i > 0$, so the total quantity produced by both companies is $q = q_1 + q_2$.

- The price of a gadget is a decreasing function of the total quantity produced by the two firms.

$$P(q) = (\Gamma - q)^+ = \begin{cases} \Gamma - q & \text{if } 0 \leq q \leq \Gamma; \\ 0 & \text{if } q > \Gamma. \end{cases}$$

Cournot Duopoly (cont'd)

- Suppose also that to make one gadget costs firm $i = 1,2$, c_i dollars per unit so the total cost to produce q_i units is $c_i q_i$, $i = 1, 2$. Assume that

$$\Gamma > c_1 + c_2,$$

  - The revenue to firm i for producing q_i units of the gadget is $q_i P(q_1 + q_2)$.
  - The cost of production to firm i is $c_i q_i$.

- Each firm wants to maximize its own profit function, which is total revenue minus total costs and is given by

$$u_1(q_1, q_2) = P(q_1 + q_2)q_1 - c_1 q_1 \text{ and } u_2(q_1, q_2) = P(q_1 + q_2)q_2 - c_2 q_2. \quad (4.2.1)$$

Cournot Duopoly (cont'd)

– We are not trying to maximize each profit function over both variables, but each profit function to each firm as a function only of the variable they control, namely, q_i.

$$\frac{\partial u_1(q_1, q_2)}{\partial q_1} = 0 \implies -2q_1 - q_2 + \Gamma - c_1 = 0,$$
$$\frac{\partial u_2(q_1, q_2)}{\partial q_2} = 0 \implies -2q_2 - q_1 + \Gamma - c_2 = 0.$$

– That is a Nash equilibrium, Now solving the resulting equations gives the optimal production quantities for each firm at

$$q_1^* = \frac{\Gamma + c_2 - 2c_1}{3} \text{ and } q_2^* = \frac{\Gamma + c_1 - 2c_2}{3}.$$

Cournot Duopoly (cont'd)

– If $\Gamma > c_1 + c_2$ then both $\Gamma > q_1^* > 0$nd $\Gamma > q_2^* > 0$At these points we have

$$\frac{\partial^2 u_1(q_1^*, q_2^*)}{\partial^2 q_1} = -2 < 0 \text{ and } \frac{\partial^2 u_2(q_1^*, q_2^*)}{\partial q_2^2} = -2 < 0,$$

and so (q_1^*, q_2^*) are values that maximize the profit functions, when the other variable is fixed.

– The total amount the two firms should produce is

$$q^* = q_1^* + q_2^* = \frac{2\Gamma - c_1 - c_2}{3}$$

– The price function at the quantity *q** is then

$$P(q_1^* + q_2^*) = \Gamma - q_1^* - q_2^* = \Gamma - \frac{2\Gamma - c_1 - c_2}{3} = \frac{\Gamma + c_1 + c_2}{3}.$$

That is the market price of the gadgets produced by both firms when producing optimally.

Cournot Duopoly (cont'd)

– Turn it around now and suppose that the price of gadgets is set at

$$P(q_1 + q_2) = p = \frac{\Gamma + c_1 + c_2}{3}.$$

If this is the market price of gadgets how many gadgets should each firm produce? The total quantity that both firms should produce (and will be sold) at this price is $q = P^{-1}(p)$,or

$$q = P^{-1}(p) = \Gamma - p = \frac{2\Gamma - c_1 - c_2}{3}$$
$$= \frac{\Gamma + c_2 - 2c_1}{3} + \frac{\Gamma + c_1 - 2c_2}{3} = q_1^* + q_2^*.$$

We conclude that the quantity of gadgets sold (demanded) will be exactly the total amount that each firm **should** produce at this price. This is called a **market equilibrium** and it turns out to be given by the Nash point equilibrium quantity to produce.

Cournot Duopoly (cont'd)

– Finally, substituting the Nash equilibrium point into the profit functions gives the equilibrium profits

$$u_1(q_1^*, q_2^*) = \frac{(\Gamma + c_2 - 2c_1)^2}{9} \text{ and } u_2(q_1^*, q_2^*) = \frac{(\Gamma + c_1 - 2c_2)^2}{9}.$$

Notice that the profit of each firm depends on the costs of the other firm. That's a problem because how is a firm supposed to know the costs of a competing firm? The costs can be estimated, but known for sure...?

A Slight Generalization of Cournot

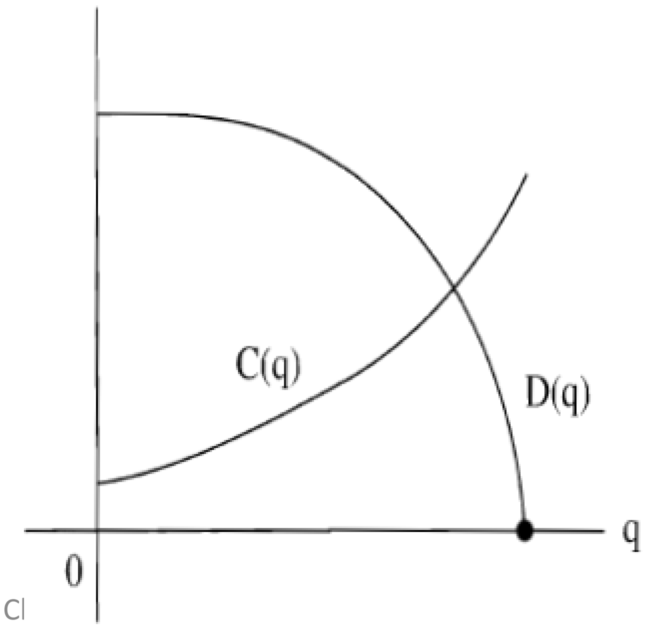
- Suppose that price is a function of the demand, which is a function of the total supply q of gadgets, so $P = D(q)$ and D is the demand function. We assume that if q gadgets are made, they will be sold at price $P = D(q)$. Suppose also that $C(z)$ is the cost to the two firms if z units of the gadget are produced.

- Again, each firm is trying to maximize its own profit, and we want to know how many gadgets each firm should produce. Another famous economist (A. Wald) solved this problem. Figure 4.2 shows the relationship of of the demand and cost functions.

A Slight Generalization of Cournot

- Suppose that price is a function of the demand, which is a function of the total supply *q* of gadgets, so $P = D(q)$ and *D* is the demand function. We assume that if *q* gadgets are made, they will be sold at price $P = D(q)$. Suppose also that $C(z)$ is the cost to the two firms if *z* units of the gadget are produced.

- Again, each firm is trying to maximize its own profit, and we want to know how many gadgets each firm should produce.



Theorem 4.2.1

- **Theorem 4.2.1**
  – *Suppose that $P = D(q)$ has two continuous derivatives, is nonin-creasing, and is concave in the interval $0 < q < \Gamma$, and suppose that*

$$D(0) > 0 \ \ and \ \ D(q) = 0, q \geq \Gamma.$$

  So there is a positive demand if there are no gadgets, but the demand (or price) shrinks to zero if too many gadgets are on the market. Also, P = D(q), the price per gadget, decreases as the total available quantity of gadgets increases. Suppose that firm i = l,2 *has $M_i > \Gamma$ gadgets available for sale.*

  Suppose that the cost function C has two continuous derivatives, is strictly in-creasing, nonnegative, and convex, and that $C'(0) < D(0)$. The payoff functions are again the total profit to each firm:

Theorem 4.2.1 (cont'd)

$$u_1(q_1, q_2) = q_1 D(q_1 + q_2) - C(q_1) \text{ and } u_2(q_1, q_2) = q_2 D(q_1 + q_2) - C(q_2).$$

Then, there is one and only one Nash equilibrium given by (q^*, q^*), *where* $q^* \in [0, \Gamma]$ is *the unique solution of the equation*

$$D(2q) + qD'(2q) - C'(q) = 0 \text{ in the interval } 0 < q < \frac{\Gamma}{2}.$$

- **Sketch of the Proof**
  - By the assumptions we put on *D* and C, we may apply the theorem that guarantees that there is a Nash equilibrium to know that we are looking for something that exists. Call it (q_1^*, q_2^*).We assume that this will happen with $0 < q_1^* + q_2^* < \Gamma$. By taking the partial derivatives and setting equal to zero, we see that

Theorem 4.2.1 (cont'd)

$$\frac{\partial u_1(q_1^*, q_2^*)}{\partial q_1} = D(q_1^* + q_2^*) + q_1^* D'(q_1^* + q_2^*) - C'(q_1^*) = 0$$

and

$$\frac{\partial u_2(q_1^*, q_2^*)}{\partial q_2} = D(q_1^* + q_2^*) + q_2^* D'(q_1^* + q_2^*) - C'(q_2^*) = 0.$$

We solve these equations by subtracting to get

$$(q_1^* - q_2^*)D'(q_1^* + q_2^*) - (C'(q_1^*) - C'(q_2^*)) = 0. \qquad (4.2.2)$$

Remember that $C'(q) > 0$ (so C' is increasing) and $D' < 0$. This means that if $q_1^* < q_2^*$ then we have the sum of two positive quantities in (4.2.2) adding to zero, which is impossible. So, it must be true that $q_1^* \geq q_2^*$. However, by a similar argument, strict inequality would be impossible, and so we conclude that $q_1^* = q_2^* = q^*$. Actually, this should be obvious because the firms are symmetric and have the same costs.

Theorem 4.2.1 (cont'd)

So now we have

$$D(2q^*) + q^* D'(q^*) - C'(q^*) = 0 \text{ and } 0 < q^* < \frac{\Gamma}{2}.$$

In addition, it is not too difficult to show that q^* is the unique root of this equation (and therefore the only stationary point). That is the place where the assumption that $C'(0) < D(0)$ is used. Finally, by taking second derivatives, we see that

$$\frac{\partial^2 u_i(q_1^*, q_2^*)}{\partial q_i^2} = 3D'(2q^*) + q^* D''(2q^*) - C''(q^*) < 0, \quad i = 1, 2,$$

for each payoff function, and hence the unique root of the equation is indeed a Nash equilibrium. □

EXAMPLE 4.3

Let's take $D(q) = 100 - q^2, 0 \leq q \leq 10$, and cost function $C(q) = 50 + q^2$. Then $C'(0) = 0 < D(0) = 100$, and the functions satisfy the appropriate convexity assumptions. Then q is the unique solution of

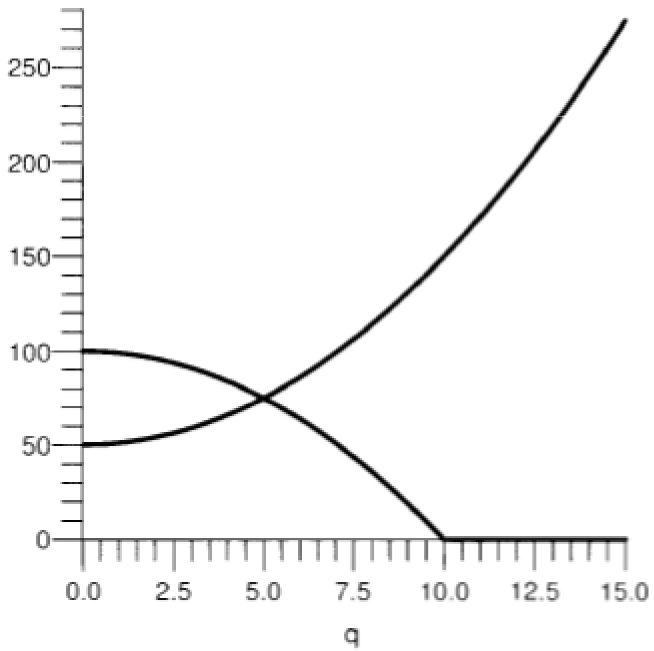
$$D(2q) + qD'(2q) - C'(q) = 100 - 8q^2 - 2q = 0$$

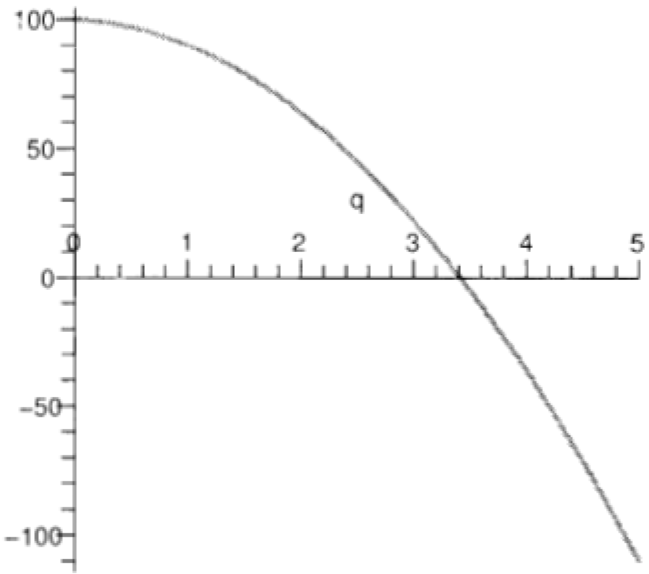
in the interval $0 < q < 10$. The unique solution is given by $q^* = 3.413$. The unit price at this quantity should be $D(2q^*) = 88.353$, and the profit to each firm will be $u_i(q^*, q^*) = q^* D(2q^*) - C(q^*) = 120.637$. These numbers, as well as some pictures, can be obtained using the Maple commands

```
> restart:
> De:=q->piecewise(q<10,100-q^2,q>=10,0);
> C:=q->50+q^2;
> plot({De(q),C(q)},q=0..15,color=[red,green]);
> diff(De(q),q);
> a:=eval( %, [q = 2*q] );
> qstar:=fsolve(De(2*q)+q*a-diff(C(q),q)=0,q);
> plot(De(2*q)+q*a-diff(C(q),q),q=0..5);
> De(qstar);
> qstar*De(2*qstar)-C(qstar);
```

EXAMPLE 4.3 (cont'd)

This will give you a plot of the demand function and cost function on the same set of axes, and then a plot of the function giving you the root where it crosses the q axis. You may modify the demand and cost functions to solve the exercises. Here are the Maple generated figures for our problem.





Cournot Model with Uncertain Costs

- Now here is a generalization of the Cournot model that is more realistic and also more difficult to solve because it involves a lack of information on the part of at least one player.
  - It is assumed that one firm has no information regarding the other firm's cost function.

- Here is the model setup.
  - Assume that both firms produce gadgets at constant unit cost.
  - Both firms know that firm 1's cost is c_1, but firm 1 does not know firm 2's cost of c_2, which is known only to firm 2.
  - Suppose that the cost for firm 2 is considered as a random variable to firm 1, say, C_2. Now firm 1 has reason to believe that

Cournot Model with Uncertain Costs (cont'd)

$$Prob(C_2 = c^+) = p \text{ and } Prob(C_2 = c^-) = 1 - p$$

for some $0 < p < 1$ that is known by firm 1.

– **Again, the payoffs to each firm are its profits.**

Firm 1's payoff function is

$$u_1(q_1, q_2) = q_1[P(q_1 + q_2) - c_1],$$

Firm 2's payoff function is

$$u_2(q_1, q_2) = q_2[P(q_1 + q_2) - C_2].$$

From firm 1 's perspective this is a random variable because of the unknown cost.

Cournot Model with Uncertain Costs (cont'd)

– The way to find an equilibrium now is the following:

1. Find the optimal production level for firm 2 using the costs c^+ and c^- giving the two numbers q_2^+ and q_2^-.
2. Firm 1 now finds the optimal production levels for the two firm 2 quantities from step 1 using the expected payoff

$$\begin{aligned}E(u_1(q_1, q_2(C_2))) &= [u_1(q_1, q_2^-)]Prob(C_2 = c^-) \\ &\quad + [u_1(q_1, q_2^+)]Prob(C_2 = c^+) \\ &= q_1[P(q_1 + q_2^-) - c_1](1 - p) + q_1[P(q_1 + q_2^+) - c_1]p.\end{aligned}$$

3. From the previous two steps you end up with three equations involving q_1, q_2^+, q_2^-. Treat these as three equations in three unknowns and solve.

Cournot Model with Uncertain Costs (cont'd)

For example, let's take the price function $P(Q) = \Gamma - Q, 0 \leq q \leq \Gamma$.

1. Firm 2 has the cost of production $c^+ q_2$ with probability p and the cost of production $c^- q_2$ with probability $1 - p$. Firm 2 will solve the problem for each cost c^+, c^- assuming that q_1 is known:

$$\max_{q_2} q_2(\Gamma - (q_1 + q_2) - c^+) \implies q_2^+ = \frac{1}{2}(\Gamma - q_1 - c^+)$$

$$\max_{q_2} q_2(\Gamma - (q_1 + q_2) - c^-) \implies q_2^- = \frac{1}{2}(\Gamma - q_1 - c^-)$$

Cournot Model with Uncertain Costs (cont'd)

2. Next, firm 1 will maximize the expected profit using the two quantities q_2^+, q_2^-. Firm 1 seeks the production quantity q_1, which solves

$$\max_{q_1} q_1[\Gamma - (q_1 + q_2^+) - c_1]p + q_1[\Gamma - (q_1 + q_2^-) - c_1](1 - p).$$

This is maximized at

$$q_1 = \frac{1}{2}[p(\Gamma - q_2^+ - c_1) + (1 - p)(\Gamma - q_2^- - c_1)].$$

Cournot Model with Uncertain Costs (cont'd)

3. Summarizing, we now have the following system of equations for the variables q_1, q_2^-, q_2^+:

$$q_2^+ = \frac{1}{2}(\Gamma - q_1 - c^+),$$
$$q_2^- = \frac{1}{2}(\Gamma - q_1 - c^-),$$
$$q_1 = \frac{1}{2}[p(\Gamma - q_2^+ - c_1) + (1-p)(\Gamma - q_2^- - c_1)].$$

Solving these, we finally arrive at the optimal production levels:

$$q_1^* = \frac{1}{3}[\Gamma - 2c_1 + pc^+ + (1-p)c^-],$$
$$q_2^{+*} = \frac{1}{3}[\Gamma + c_1] - \frac{1}{6}[(1-p)c^- + pc^+] - \frac{1}{2}c^+,$$
$$q_2^{-*} = \frac{1}{3}[\Gamma - 2c^- + c_1] + \frac{1}{6}p(c^- - c^+).$$

The Bertrand Model

- Here is the setup. We again have two companies making identical gadgets. *In this model they can set prices, not quantities, and they will only produce the quantity demanded at the given price.* So the quantity sold is a function of the price set by each firm, say, $q = \Gamma - p$. This is better referred to as the **demand function** for a given price:

$$D(p) = \Gamma - p, \quad 0 \le p \le \Gamma \quad \text{and} \quad D(p) = 0 \text{ when } p > \Gamma.$$

The Bertrand Model (cont'd)

- In a classic problem the model says that if both firms charge the same price, they will split the market evenly, with each selling exactly half of the total sold. But the company that charges a lower price will capture the entire market.
  - We have to assume that each company has enough capacity to make the entire quantity if it captures the whole market.

- The cost to make gadgets is still c_i , i = 1,2, dollars per unit gadget. We first assume:

$$c_1 \neq c_2 \text{ and } \max\{c_1, c_2\} < \Gamma + \min\{c_1, c_2\}.$$

The Bertrand Model (cont'd)

- The profit function for firm *i* = 1,2, assuming that firm 1 sets the price as p_1 and firm 2 sets the price at p_2, is

$$u_1(p_1,p_2) = \begin{cases} p_1(\Gamma - p_1) - c_1(\Gamma - p_1) & \text{if } p_1 < p_2; \\ \dfrac{(p - c_1)(\Gamma - p)}{2} & \text{if } p_1 = p_2 = p \geq c_1; \\ 0, & \text{if } p_1 > p_2. \end{cases}$$

$$u_2(p_1,p_2) = \begin{cases} p_2(\Gamma - p_2) - c_2(\Gamma - p_2) & \text{if } p_2 < p_1; \\ \dfrac{(p - c_2)(\Gamma - p)}{2} & \text{if } p_1 = p_2 = p \geq c_2; \\ 0 & \text{if } p_2 > p_1. \end{cases}$$

The Bertrand Model (cont'd)

- So now we have to find a Nash equilibrium. Let's suppose that there is a Nash equilibrium point at (p_1^*, p_2^*) - By definition, we have

$$u_1(p_1^*, p_2^*) \geq u_1(p_1, p_2^*) \text{ and } u_2(p_1^*, p_2^*) \geq u_2(p_1^*, p_2), \text{ for all } (p_1, p_2).$$

Case 1. $p_1^* > p_2^*$. Then it should be true that firm 2, having a lower price, captures the entire market so that for firm 1

$$u_1(p_1^*, p_2^*) = 0 \geq u_1(p_1, p_2^*), \text{ for every } p_1.$$

But if we take any price $c_1 < p_1 < p_2^*$, the right side will be

$$u_1(p_1, p_2^*) = (p_1 - c_1)(\Gamma - p_1) > 0,$$

so $p_1^* > p_2^*$ cannot hold and still have (p_1^*, p_2^*) a Nash equilibrium.

The Bertrand Model (cont'd)

Case 2. $p_1^* < p_2^*$. Then it should be true that firm 1 captures the entire market and so for firm 2

$$u_2(p_1^*, p_2^*) = 0 \geq u_2(p_1^*, p_2), \text{ for every } p_2.$$

But if we take any price for firm 2 with $c_2 < p_2 < p_1^*$, the right side will be

$$u_2(p_1^*, p_2) = (p_2 - c_2)(\Gamma - p_2) > 0,$$

that is, strictly positive, and again it cannot be that $p_1^* < p_2^*$ and fulfill the requirements of a Nash equilibrium. So the only case left is the following.

The Bertrand Model (cont'd)

Case 3. $p_1^* = p_2^*$. But then the two firms split the market and we must have for firm 1

$$u_1(p_1^*, p_2^*) = \frac{(p_1^* - c_1)(\Gamma - p_1^*)}{2} \geq u_1(p_1, p_2^*), \text{ for all } p_1 \geq c_1.$$

If we take firm 1's price to be $p_1 = p_1^* - \varepsilon < p_2^*$ with really small $\varepsilon > 0$, then firm 1 drops the price ever so slightly below firm 2's price. Under the Bertrand model, firm 1 will capture the entire market at price p_1 so that in this case we have

$$u_1(p_1^*, p_2^*) = \frac{(p_1^* - c_1)(\Gamma - p_1^*)}{2} < u_1(p_1^* - \varepsilon, p_2^*) = (p_1^* - \varepsilon - c_1)(\Gamma - p_1^* + \varepsilon).$$

This inequality won't be true for every ε, but it will be true for small enough $\varepsilon > 0$, (say, $0 < \varepsilon < (p_1^* - c_1)/2$).

- So, in all cases, we can find prices so that the condition that (p_1^*, p_2^*) be a Nash point is violated and so ***there is no Nash equilibrium in pure strategies.***

The Bertrand Model (cont'd)

- But there is one case when there is a Nash equilibrium. In the analysis above we assumed in several places that prices would have to be above costs. What if we drop that assumption?
  - In fact, we are led to believe that maybe $p_1^* = c_1, p_2^* = c_2$ is a Nash equilibrium. Let's check that, and let's just assume that $c_1 < c_2$, because a similar argument would apply if $c_1 > c_2$.

- In this case, $u_1(c_1, c_2) = 0$, and if this is a Nash equilibrium, then it must be true that $u_1(c_1, c_2) = 0 \geq u_1(p_1, c_2)$ for all p_1. But if we take any price $c_1 < p_1 < c_2$, then $u_1(p_1, c_2) = (p_1 - c_1)(\Gamma - p_1) > 0,$ and we conclude that (c_1, c_2) also is not a Nash equilibrium.

The Bertrand Model (cont'd)

- The only way that this could work is if $c_1 = c_2 = c$. so the costs to each firm are the same. In this case we leave it as an exercise to show that $p_1^* = c, p_2^* = c$ is a Nash equilibrium and optimal profits are zero for each firm.

- So, what good is this if the firms make no money, and even that is true only when their costs are the same?
  - This leads us to examine assumptions about exactly how profits arise in competing firms.
  - Is it strictly prices and costs, or are there other factors involved ?

EXAMPLE 4.4

The Traveler's Paradox. Two airline passengers who have luggage with identical contents are informed by the airline that their luggage has been lost. The airline offers to compensate them if they make a claim in some range acceptable to the airline. Here are the payoff functions for each player

$$u_1(q_1, q_2) = \begin{cases} q_1 & \text{if } q_1 = q_2; \\ q_1 + R & \text{if } q_2 > q_1; \\ q_2 - R & \text{if } q_1 > q_2. \end{cases} \quad \text{and} \quad u_2(q_1, q_2) = \begin{cases} q_2 & \text{if } q_1 = q_2; \\ q_1 - R & \text{if } q_2 > q_1; \\ q_2 + R & \text{if } q_1 > q_2. \end{cases}$$

It is assumed that the acceptable range is $[a, b]$ and $q_i \in [a, b], i = 1, 2$. The idea behind these payoffs is that if the passengers' claims are equal the airline will pay the amount claimed. If passenger I claims less than passenger II, $q_1 < q_2$, then passenger II will be penalized an amount R and passenger I will receive the amount she claimed plus R. Passenger II will receive the lower amount claimed minus R. Similarly, if passenger I claims more than does passenger II, $q_1 > q_2$, then passenger I will receive $q_2 - R$, and passenger II will receive the amount claimed q_2 plus R.

EXAMPLE 4.4 (cont'd)

Suppose, to be specific, that $a = 80, b = 200$, so the airline acceptable range is from \$80 to \$200. We take $R = 2$, so the penalty is only 2 dollars for claiming high. Believe it or not, we will show that the Nash equilibrium is $(q_1 = 80, q_2 = 80)$ so both players should claim the low end of the airline range (under the Nash equilibrium concept). To see that, we have to show that

$$u_1(80, 80) = 80 \geq u_1(q_1, 80) \quad \text{and} \quad u_2(80, 80) = 80 \geq u_2(80, q_2)$$

for all $80 \leq q_1, q_2 \leq 200$. Now

$$u_1(q_1, 80) = \begin{cases} 80 & \text{if } q_1 = 80; \\ 80 - 2 = 78 & \text{if } q_1 > 80. \end{cases}$$

and

$$u_2(80, q_2) = \begin{cases} 80 & \text{if } q_2 = 80; \\ 80 - 2 = 78 & \text{if } q_2 > 80; \end{cases}$$

EXAMPLE 4.4 (cont'd)

So indeed $(80, 80)$ is a Nash equilibrium with payoff 80 to each passenger. But clearly, they can do better if they both claim $q_1 = q_2 = \$200$. Why don't they do that? The problem is that there is an incentive to undercut the other traveler. If R is \$2, then, if one of the passengers drops her claim to \$199, this passenger will actually receive $\$201$. This cascades downward, and the undercutting disappears only at the lowest range of the acceptable claims. Do you think that the passengers would, in reality, make the lowest claim?

The Stackelberg Model

- *What happens if two competing firms don't choose the production quantities at the same time, but one after the other?* Stackelberg gave an answer to this question.
  - In this model we will assume that there is a dominant firm, say, firm 1, who will announce its production quantity publicly. Then firm 2 will decide how much to produce.

- Suppose that firm 1 announces that it will produce q_1 gadgets at cost c_1 dollars per unit. It is then up to firm 2 to decide how many gadgets, say, q_2 at cost c_2 it will produce.

The Stackelberg Model (cont'd)

- The price per unit will then be considered a function of the total quantity produced so that

$$p = p(q_1, q_2) = (\Gamma - q_1 - q_2)^+ = \max\{\Gamma - q_1 - q_2, 0\}$$

- The profit functions will be

$$u_1(q_1, q_2) = (\Gamma - q_1 - q_2)q_1 - c_1 q_1,$$
$$u_2(q_1, q_2) = (\Gamma - q_1 - q_2)q_2 - c_2 q_2.$$

  – These are the same as in the simplest Cournot model.

The Stackelberg Model (cont'd)

- So what we are really looking for is the best response of firm 2 to the production announcement q_1 by firm 1.
  - In other words, firm 2 wants to know how to choose $q_2 = q_2(q_1)$ so as to

$$\text{Maximize over } q_2 \text{, given } q_1 \text{, the function } u_2(q_1, q_2(q_1)).$$

  - This is given by calculus as

$$q_2(q_1) = \frac{\Gamma - q_1 - c_2}{2}.$$

    This is the amount that firm 2 should produce when firm 1 announces the quantity of production q_1.

The Stackelberg Model (cont'd)

- Firm 1 knows what firm 2's optimal production quantity should be, given its own announcement of q_1.
  - Therefore, firm 1 should choose q_1 to maximize its own profit function knowing that firm 2 will use production quantity $q_2(q_1)$:

$$\begin{aligned} u_1(q_1, q_2(q_1)) &= q_1(\Gamma - q_1 - q_2(q_1)) - c_1 q_1 \\ &= q_1\left(\Gamma - q_1 - \frac{\Gamma - q_1 - c_2}{2}\right) - c_1 q_1 \\ &= q_1 \frac{\Gamma - q_1}{2} + q_1\left(\frac{c_2}{2} - c_1\right). \end{aligned}$$

  - Firm 1 wants to choose q_1 to make this as large as possible. By calculus, we find that

$$q_1^* = \frac{\Gamma - 2c_1 + c_2}{2}, \quad \text{and} \quad u_1(q_1^*, q_2^*) = \left(\frac{\Gamma - 2c_1 + c_2}{2}\right)^2.$$

The Stackelberg Model (cont'd)

- Then the optimal production quantity for firm 2 will be

$$q_2^* = q_2(q_1^*) = \frac{\Gamma + 2c_1 - 3c_2}{4}.$$

- The equilibrium profit function for firm 2 is then

$$u_2(q_1^*, q_2^*) = \frac{(\Gamma + 2c_1 - 3c_2)^2}{16},$$

and for firm 1, it is

$$u_1(q_1^*, q_2^*) = \left(\frac{\Gamma - 2c_1 + c_2}{2}\right)^2.$$

The Stackelberg Model (cont'd)

- Setting $c_1 = c_2 = c$ and then recall the optimal production quantities for the Cournot model:

$$q_1^c = \frac{\Gamma - 2c_1 + c_2}{3} = \frac{\Gamma - c}{3}, \qquad q_2^c = \frac{\Gamma + c_1 - 2c_2}{3} = \frac{\Gamma - c}{3}.$$

- The equilibrium profit functions were

$$u_1(q_1^c, q_2^c) = \frac{(\Gamma + c_2 - 2c_1)^2}{9} = \frac{(\Gamma - c)^2}{9},$$

$$u_2(q_1^c, q_2^c) = \frac{(\Gamma + c_1 - 2c_2)^2}{9} = \frac{(\Gamma - c)^2}{9}.$$

- In the Stackelberg model we have

$$q_1^* = \frac{\Gamma - c}{2} > q_1^c, \quad q_2^* = \frac{\Gamma - c}{4} < q_2^c.$$

The Stackelberg Model (cont'd)

- So firm 1 produces more and firm 2 produces less in the Stackelberg model than if firm 2 did not have the information announced by firm 1.

$$u_1(q_1^c, q_2^c) = \frac{(\Gamma - c)^2}{9} < u_1(q_1^*, q_2^*) = \frac{(\Gamma - c)^2}{4},$$

$$u_2(q_1^c, q_2^c) = \frac{(\Gamma - c)^2}{9} > u_2(q_1^*, q_2^*) = \frac{(\Gamma - c)^2}{16}.$$

  - **Firm 1 makes more money by announcing the production level, and firm 2 makes less with the information.**

The Stackelberg Model (cont'd)

- One last comparison is the total quantity produced

$$q_1^c + q_2^c = \frac{2}{3}(\Gamma - c) < q_1^* + q_2^* = \frac{3\Gamma - 2c_1 - c_2}{4} = \frac{3}{4}(\Gamma - c)$$

and the price at equilibrium

$$P(q_1^* + q_2^*) = \frac{\Gamma + 3c}{4} < P(q_1^c + q_2^c) = \frac{\Gamma + 2c}{3}.$$

Entry Deterrence

- If there is currently only one firm producing a gadget, what should be the price of the gadget in order to make it unprofitable for another firm to enter the market and compete with firm 1? This is a famous problem in economics called the **entry deterrence problem.**

- The existing company: firm 1
- The potential challenger: firm 2
- The demand function: $p = D(q) = (\Gamma - q)^+$

Entry Deterrence (cont'd)

- Now, before the challenger enters the market the profit function to firm 1 is

$$u_1(q_1) = (\Gamma - q_1)q_1 - (aq_1 + b)$$

cost function

$$C(q) = aq + b, \text{ with } \Gamma > a, b > 0$$

  - This cost function includes a fixed cost of *b* > 0 because even if the firm produces nothing, it still has expenses.

Entry Deterrence (cont'd)

- Now firm 1 is acting as a monopolist in our model because it has no competition.
  - So firm 1 wants to maximize profit, which gives a production quantity of

$$q_1^* = \frac{\Gamma - a}{2}$$

  - and maximum profit for a monopolist of

$$u_1(q_1^*) = \frac{(\Gamma - a)^2}{4} - b.$$

  - the price of a gadget at this quantity of production will be

$$p = D(q_1^*) = \frac{\Gamma + a}{2}.$$

Entry Deterrence (cont'd)

- Now firm 2 enters the picture and calculates firm 2's profit function knowing that firm 1 will or should produce $q_1^* = (\Gamma - a)/2$ to get firm 2's payoff function

$$u_2(q_2) = \left(\Gamma - \frac{\Gamma - a}{2} - q_2\right) q_2 - (aq_2 + b)$$

  - So firm 2 calculates its maximum possible profit and optimal production quantity as

$$u_2(q_2^*) = \frac{(\Gamma - a)^2}{16} - b, \quad q_2^* = \frac{\Gamma - a}{4}.$$

  - The price of gadgets will now drop to

$$p = D(q_1^* + q_2^*) = \Gamma - q_1^* - q_2^* = \frac{\Gamma + 3a}{4} < D(q_1^*) = \frac{\Gamma + a}{2}.$$

Entry Deterrence (cont'd)

- As long as $u_2(q_2^*) \geq 0$, firm 2 has an incentive to enter the market. If we interpret the constant 6 as a fixed cost to enter the market, this will require that

$$\frac{(\Gamma - a)^2}{16} > b$$

- Firm 1 is not about to sit by idly and let another firm enter the market. Therefore, firm 1 will now analyze the Cournot model assuming that there is a firm 2 against which firm 1 is competing.
  - Firm 1 looks at the profit function for firm 2:

$$u_2(q_1, q_2) = (\Gamma - q_1 - q_2)q_2 - (aq_2 + b)$$

Entry Deterrence (cont'd)

and maximizes this as a function of q_2 to get the maximum profit to firm 2 if firm 1 produces q_1 gadgets.

$$q_2^m = \frac{\Gamma - q_1 - a}{2} \quad \text{and} \quad u_2(q_1, q_2^m) = \frac{(\Gamma - q_1 - a)^2}{4} - b$$

– Firm 1 reasons that it can set q_1 so that firm 2's profit is zero:

$$u_2(q_1, q_2^m) = \frac{(\Gamma - q_1 - a)^2}{4} - b = 0 \implies q_1^0 = \Gamma - 2\sqrt{b} - a.$$

Consequently, if firm 1 decides to produce gf gadgets, firm 2 has no incentive to enter the market.

Entry Deterrence (cont'd)

– The price at this quantity will be

$$D(q_1^0) = \Gamma - (\Gamma - 2\sqrt{b} - a) = 2\sqrt{b} + a,$$

and the profit for firm 1 at this level of production will be

$$u_1(q_1^0) = (\Gamma - q_1)q_1 - (aq_1 + b) = 2\sqrt{b}(\Gamma - a) - 5b.$$

This puts a requirement on Γ that $\Gamma > a + \frac{5}{2}\sqrt{b}$, or else firm 1 will also make a zero profit.

4.3 Duels Problems

Duels

- Duels are used to model not only the actual dueling situation but also many problems in other fields. For example, a battle between two companies for control of a third company or asset can be regarded as a duel in which the accuracy functions could represent the probability of success.

- Duels can be used to model competitive auctions between two bidders. So there is ample motivation to study a theory of duels.

Duels (cont'd)

- In earlier chapters we considered discrete versions of a duel in which the players were allowed to fire only at certain distances.
  - In reality, a player can shoot at any distance (or time) once the duel begins. That was only one of our simplifications.
  - The theory of duels includes multiple bullets, machine gun duels, silent and noisy, and so on.

Duels (cont'd)

- Here are the precise rules that we use here. There are two participants, I and II, each with a gun, and each has exactly one bullet. They will fire their guns at the opponent at a moment of their own choosing.

- The players each have functions representing their accuracy or probability of killing the opponent, say, $p_{\mathrm{I}}(x)$ for player I and $p_{\mathrm{II}}(y)$ for player II, with $x, y \in [0,1]$.
  - The choice of strategies is a **time** in [0,1] at which to shoot. Assume that $p_{\mathrm{I}}(0) = p_{\mathrm{II}}(0) = 0$ and $p_{\mathrm{I}}(1) = p_{\mathrm{II}}(1) = 1.$

Duels (cont'd)

- So, in the setup here you may assume that they are farthest apart at time 0 or *x* = *y* = 0 and closest together when *x* = *y* = 1.
- It is realistic to assume also that both p_{I} and p_{II} are continuous, are strictly increasing, and have continuous derivatives up to any order needed.

- Finally, if I hits II, player I receives +1 and player II receives - 1, and conversely. If both players miss, the payoff is 0 to both.

Duels (cont'd)

- The payoff functions will be the expected payoff depending on the accuracy functions and the choice of the $x \in [0, 1]$ or $y \in [0, 1]$ at which the player will take the shot.

- We break our problem down into the cases where player I shoots before player II, player II shoots before player I, or they shoot at the same time.

Duels (cont'd)

– If player I shoots before player II, then $x < y$, and

$$u_1(x,y) = (+1)p_{\mathrm{I}}(x) + (-1)(1 - p_{\mathrm{I}}(x))p_{\mathrm{II}}(y),$$
$$u_2(x,y) = (-1)p_{\mathrm{I}}(x) + (+1)(1 - p_{\mathrm{I}}(x))p_{\mathrm{II}}(y).$$

– If player II shoots before player I, then $y < x$ and we have similar expected payoffs:

$$u_1(x,y) = (-1)p_{\mathrm{II}}(y) + (+1)(1 - p_{\mathrm{II}}(y))p_{\mathrm{I}}(x),$$
$$u_2(x,y) = (+1)p_{\mathrm{II}}(y) + (-1)(1 - p_{\mathrm{II}}(y))p_{\mathrm{I}}(x).$$

– Finally, if they choose to shoot at the same time, then $x = y$ and we have

$$u_1(x,x) = (+1)p_{\mathrm{I}}(x)(1 - p_{\mathrm{II}}(x)) + (-1)(1 - p_{\mathrm{I}}(x))p_{\mathrm{II}}(x) = -u_2(x,x).$$

- In this simplest setup, this is a zero sum game, but, as mentioned earlier, it is easily changed to nonzero sum.

Duels (cont'd)

- We have set up this duel without consideration yet that the duel is noisy.
  - Each player will hear (or see, or feel) the shot by the other player, so that if a player shoots and misses, the surviving player will know that all she has to do is wait until her accuracy reaches 1. With certainty that occurs at $x = y = 1$, and she will then take her shot.
  - In a **silent** duel the players would not know that a shot was taken (unless they didn't survive). Silent duels are more difficult to analyze.

Duels (cont'd)

- Let's simplify the payoffs in the case of a noisy duel.
  - In that case, when a player takes a shot and misses, the other player (if she survives) waits until time 1 to kill the opponent with certainty.
  - So, the payoffs become

$$u_1(x,y) = \begin{cases} (+1)p_{\mathrm{I}}(x) + (-1)(1 - p_{\mathrm{I}}(x)) = 2p_{\mathrm{I}}(x) - 1, & x < y; \\ p_{\mathrm{I}}(x) - p_{\mathrm{II}}(x), & x = y; \\ 1 - \boxed{2p_{\mathrm{II}}(y)} & x > y. \end{cases}$$

For player II, $u_2(x,y) = -u_1(x,y)$.

Now, to solve this, we cannot use the procedure outlined using derivatives, because this function has no derivatives exactly at the places where the optimal things happen. ($p_{\mathrm{I}}(x)$ and $p_{\mathrm{II}}(x)$ are strictly increasing)

Duels (cont'd)

- Figure 4.4 below shows a graph of $u_1(x, y)$ in the case when the players have the distinct accuracy functions given by $p_{\mathrm{I}}(x) = x^3$ and $p_{\mathrm{II}}(x) = x^2$. Player I's accuracy function increases at a slower rate than that for player II.

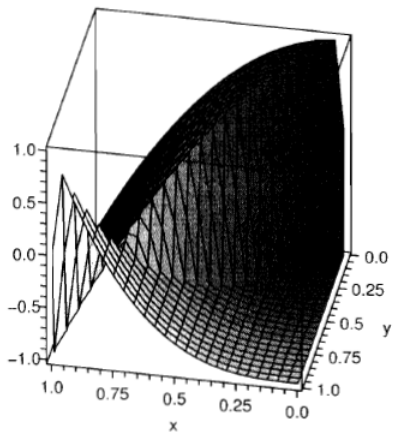


Figure 4.4 $u_1(x, y)$ with accuracy functions $p_{\mathrm{I}} = x^3, p_{\mathrm{II}} = x^2$.

Duels (cont'd)

– Nevertheless, we will see that both players will fire at the same time. That conclusion seems reasonable when it is a noisy duel. If one player fires before the opponent, the accuracy suffers, and, if it is a miss, death is certain.
– In fact, we will show that there is a unique point $x^* \in [0,1]$ that is the unique solution of

$$p_{\mathrm{I}}(x^*) + p_{\mathrm{II}}(x^*) = 1, \tag{4.3.1}$$

so that

$$u_1(x^*, x^*) \geq u_1(x, x^*) \text{ for all } x \in [0,1] \tag{4.3.2}$$

and

$$u_2(x^*, x^*) \geq u_2(x^*, y) \text{ for all } y \in [0,1]. \tag{4.3.3}$$

This says that (x^*, x^*) is a Nash equilibrium for the noisy duel.

Duels (cont'd)

– Of course, since $u_2 = -u_1$, the inequalities reduce to

$$u_1(x, x^*) \leq u_1(x^*, x^*) \leq u_1(x^*, y), \text{ for all } x, y \in [0, 1],$$

so that (x^*, x^*) is a saddle point for u_1 .

– To verify that the inequalities (4.3.2) and (4.3.3) hold for x* defined in (4.3.1), we have, from the definition *of* u_1*,* that

$$u_1(x^*, x^*) = p_{\mathrm{I}}(x^*) - p_{\mathrm{II}}(x^*).$$

Duels (cont'd)

– Using the fact that both accuracy functions are increasing, we have, by (4.3.1)

$$\begin{aligned} u_1(x^*, x^*) &= p_{\mathrm{I}}(x^*) - p_{\mathrm{II}}(x^*) = 1 - 2p_{\mathrm{II}}(x^*) \\ &\leq 1 - 2p_{\mathrm{II}}(y) = u_1(x^*, y) \text{ if } x^* > y, \\ u_1(x^*, x^*) &= p_{\mathrm{I}}(x^*) - p_{\mathrm{II}}(x^*) \\ &= 2p_{\mathrm{I}}(x^*) - 1 = u_1(x^*, y) \text{ if } x^* < y, \quad \text{and} \\ u_1(x^*, x^*) &= p_{\mathrm{I}}(x^*) - p_{\mathrm{II}}(x^*) \\ &= u_1(x^*, y) \text{ if } x^* = y. \end{aligned}$$

So, in all cases $u_1(x^*, x^*) \leq u_1(x^*, y)$ for all $y \in [0, 1]$.

– We can verify (4.3.3) in a similar way.

$$u_2(x^*, x^*) \geq u_2(x^*, y) \text{ for all } y \in [0, 1]. \qquad (4.3.3)$$

Duels (cont'd)

- That x^* exists and is unique is shown by considering the function

  $f(x) = p_{\mathrm{I}}(x) + p_{\mathrm{II}}(x)$.

  We have $f(0) = 0, f(1) = 2$, and $f'(x) = p'_{\mathrm{I}}(x) + p'_{\mathrm{II}}(x) > 0$. By the **intermediate value theorem** of calculus we conclude that there is an x^* satisfying $f(x^*) = 1$. The uniqueness of x^* follows from the fact that p_{I} and p_{II} are strictly increasing.

- In the example shown in Figure 4.4 with $p_{\mathrm{I}}(x) = x^3$ and $p_{\mathrm{II}}(x) = x^2$, we have the condition $x^{*3} + x^{*2} = 1$, which has solution at $x^* = 0.754877$. With these accuracy functions, the duelists should wait until less than 25% of the time is left until they **both** fire. The expected payoff to player I is $u_1(x^*, x^*) = -0.1397$, and the expected payoff to player II is $u_2(x^*, x^*) = 0.1397$. It appears that player I is going down. We expect that result in view of the fact that player II has greater accuracy at x^*, namely, $p_{\mathrm{II}}(x^*) = 0.5698 > p_{\mathrm{I}}(x^*) = 0.4301$.

Duels (cont'd)

- The following Maple commands are used to get all of these results and see some great pictures:

```
> restart:with(plots):
> p1:=x->x^3;p2:=x->x^2;
> v1:=(x,y)->piecewise(x<y,2*p1(x)-1,x=y,p1(x)-p2(x),x>y,1-2*p2(x));
> plot3d(v1(x,y),x=0..1,y=0..1,axes=boxed);
> xstar:=fsolve(p1(x)+p2(x)-1=0,x);
> v1(xstar,xstar);
```

Silent Duels on [0, 1]

- In case you are curious as to what happens when we have a silent duel, we will present this example to show that things get considerably more complicated.
  - We take the simplest possible accuracy functions $p_{\mathrm{I}}(x) = p_{\mathrm{II}}(x) = x \in [0, 1]$ because this case is already much more difficult than the noisy duel. The payoff of this game to player I is

$$u_1(x, y) = \begin{cases} x - (1-x)y, & x < y; \\ 0, & x = y; \\ -y + (1-y)x, & x > y. \end{cases}$$

    For player II, since this is zero sum, $u_2(x, y) = -u_1(x, y)$.

Silent Duels on [0, 1] (cont'd)

- Now, in the problem with a silent duel, intuitively it seems that there cannot be a pure Nash equilibrium because silence would dictate that an opponent could always take advantage of a pure strategy. But how do we allow mixed strategies in a game with continuous strategies?
  - In a discrete matrix game a mixed strategy is a probability distribution over the pure strategies. Why not allow the players to choose continuous probability distributions? No reason at all. So we consider the mixed strategy choice for each player

Silent Duels on [0, 1] (cont'd)

$$X(x) = \int_0^x f(a)\,da,$$
$$Y(y) = \int_0^y g(b)\,db,$$
$$\int_0^1 f(a)\,da = \int_0^1 g(b)\,db = 1.$$

– The cumulative distribution function $X(x)$ represents the probability that player I will choose to fire at a point $\leq x$.

Silent Duels on [0, 1] (cont'd)

– The expected payoff to player I if he chooses *X* and his opponent chooses *Y* is

$$E(u_1(X,Y)) = \int_0^1 \int_0^1 u_1(x,y) f(x) g(y)\ dx\, dy.$$

– As in the discrete-game case, we define the value of the game as

$$v \equiv \min_Y \max_X E(u_1(X,Y)) = \max_X \min_Y E(u_1(X,Y)).$$

- The equality follows from the existence theorem of a Nash equilibrium (actually a saddle point in this case) because the expected payoff is not only concave-convex, but actually linear in each of the probability distributions *X*, *Y*. It is completely analogous to the existence of a mixed strategy saddle point for matrix games.

Silent Duels on [0, 1] (cont'd)

– A saddle point in mixed strategies has the same definition as before: (X^*, Y^*) is a saddle if

$$E(X, Y^*) \le E(X^*, Y^*) = v \le E(X^*, Y), \forall X, Y \text{ probability distributions.}$$

– Now, the fact that both players are symmetric and have the same accuracy functions allows us to guess that *v* = 0 for the silent duel.
  - To find the optimal strategies, namely, the density functions $f(x), g(y)$, we use the necessary condition that if X^*, Y^* are optimal, then

$$E(X, y) = \int_0^1 u_1(x, y) f(x)\, dx = v = 0,$$
$$E(x, Y) = \int_0^1 u_1(x, y) g(y)\, dy = v = 0, \quad \forall x, y \in [0, 1].$$

Silent Duels on [0, 1] (cont'd)

This is completely analogous to the equality of payoffs Theorem 3.2.4 to find mixed strategies in bimatrix games, or to the geometric solution of two person 2 x 2 games in which the value occurs where the two payoff lines cross.

- We replace $u_1(x, y)$ to work with the following equation:

$$\int_0^y [x - (1-x)y] f(x)\, dx + \int_y^1 [-y + (1-y)x] f(x)\, dx = 0, \ \forall\, y \in [0, 1].$$

If we expand this, we get

$$\begin{aligned} 0 &= \int_0^y [x - (1-x)y] f(x)\, dx + \int_y^1 [-y + (1-y)x] f(x)\, dx \\ &= \int_0^y x f(x)\, dx - y \int_0^y (1-x) f(x)\, dx - y \int_y^1 f(x) dx + (1-y) \int_y^1 x f(x)\, dx \\ &= \int_0^1 x f(x)\, dx - y \int_0^1 f(x)\, dx + y \int_0^y x f(x)\, dx - y \int_y^1 x f(x)\, dx \\ &= \int_0^1 x f(x)\, dx - y + y \int_0^y x f(x)\, dx - y \int_y^1 x f(x)\, dx. \end{aligned}$$

mean of the strategy X
$E[X]$

Silent Duels on [0, 1] (cont'd)

- Now a key observation is that the equation we have should be looked at, not in the unknown function $f(x)$, but in the unknown function $xf(x)$. Let's call it $\varphi(x) \equiv xf(x)$, and we see that

$$E[X] - y + y \int_0^y \varphi(x)\ dx - y \int_y^1 \varphi(x)\ dx = 0.$$

Consider the left side as a function of $y \in [0, 1]$. Call it

$$F(y) \equiv E[X] - y + y \int_0^y \varphi(x)\ dx - y \int_y^1 \varphi(x)\ dx.$$

Then $F(y) = 0, 0 \le y \le 1$. We take a derivative using the fundamental theorem of calculus in an attempt to get rid of the integrals:

$$F'(y) = -1 + \int_0^y \varphi(x)\ dx + y[\varphi(y)] - \int_y^1 \varphi(x)\ dx + y[\varphi(y)] = 0$$

Silent Duels on [0, 1] (cont'd)

and then another derivative

$$F''(y) = \varphi(y) + \varphi(y) + y\varphi'(y) + \varphi(y) + \varphi(y) + y\varphi'(y)$$
$$= 4\varphi(y) + 2y\varphi'(y) = 0.$$

So we are led to the differential equation for $\varphi(y)$, which is

$$4\varphi(y) + 2y\varphi'(y) = 0, \quad 0 \le y \le 1.$$

This is a first-order **ordinary differential equation** that will have general solution

$$\varphi(y) = C\frac{1}{y^2},$$

as you can easily check by plugging in. $\varphi(y) = yf(y) = C/y^2$ implies that $f(y) = C/y^3$, or $f(x) = C/x^3$ returning to the *x* variable.

Silent Duels on [0, 1] (cont'd)

- We have to determine the constant *C.*

  You might think that the way to find *C* is to apply the fact that $\int_0^1 f(x)\ dx = 1$. That would normally be correct, but it also points out a problem with our formulation. Look at $\int_0^1 x^{-3}\ dx$. This integral **diverges** (that is, it is infinite), because x^{-3} is not integrable on (0,1). This would stop us dead in our tracks because there would be no way to fix that with a constant *C* unless the constant were zero. That can't be, because then $f = 0$, and it is not a probability density.

- The way to fix this is to assume that the function *f(x)* is zero on the starting subinterval [0, a] for some 0 < *a* < 1. In other words, we are assuming that the players will not shoot on the interval [0, a] for some unknown time a > 0.

Silent Duels on [0, 1] (cont'd)

- The lucky thing is that the procedure we used at first, but now repeated with this assumption, is the same and leads to the equation

$$E[X] - y + y\int_a^y \varphi(x)\,dx - y\int_y^1 \varphi(x)\,dx = 0,$$

which is the same as where we were before except that 0 is replaced by a. So, we get the same function $\varphi(y)$ and eventually the same $f(x) = C/x^3$, except we are now on the interval $0 < a \le x \le 1$.

- This idea does not come for free, however, because now we have two constants to determine, *C* and *a. C* is easy to find because we must have $\int_a^1 C/x^3\,dx = 1$.

- This says, $C = 2a^2/(l - a^2) > 0$. To find $a > 0$, we substitute $f'(x) = C/x^3$ into (recall that $\varphi(x) = x\,f(x)$)

$$\begin{aligned}0 &= E[X] - y + y\int_0^y \varphi(x)\,dx - y\int_y^1 \varphi(x)\,dx\\ &= y\left(C + \frac{C}{a} - 1\right) + C\left(-3 + \frac{1}{a}\right).\end{aligned}$$

Silent Duels on [0, 1] (cont'd)

- This must hold for all $a \le y \le 1$ which implies that *C* +*C/a* —1 = 0. Therefore, *C* = *a/*(*a* + 1). But then we must have

$$C = \frac{2a^2}{1-a^2} = \frac{a}{a+1} \implies a = \frac{1}{3}, \quad C = \frac{1}{4}.$$

- So, we have found $X(x)$. It is the cumulative distribution function of the strategy for player I and has density

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < \frac{1}{3} \\ 1/(4x^3) & \text{if } \frac{1}{3} \le x \le 1 \end{cases}$$

Silent Duels on [0, 1] (cont'd)

- We know that $\int_a^1 u_1(x,y)f(x)\,dx = 0$ for $y \geq a$, but we have to check that with this *C* = 1/4 and *a* = 1/3 to make sure that

$$\int_a^1 u_1(x,y)f(x)\,dx > 0 = v, \text{ when } y < a. \qquad (4.3.4)$$

That is, we need to check that *X* played against any pure strategy in [0, a] must give at least the value *v* = 0 if *X* is optimal. Let's take a derivative of the function $G(y) = \int_a^1 u_1(x,y)f(x)\,dx,\, 0 \leq a \leq 1$. We have,

$$G(y) = \int_a^1 u_1(x,y)f(x)\,dx = \int_a^1 (-y + (1-y)x)f(x)\,dx,$$

which implies that

$$G'(y) = \int_a^1 [-1 - xf(x)]\,dx = -\frac{3}{2} < 0.$$

Silent Duels on [0, 1] (cont'd)

So, $G(y)$ is decreasing on [0, a]. Since $G(a) = \int_a^1 (-a + (1-a)x) f(x)\ dx = 0$, it must be true that $G(y)$ > 0 on [0, a), so the condition (4.3.4) checks out. Finally, since this is a symmetric game, it will be true that $Y(y)$ will have the same density as player I.

4.4 Auctions

Definition 4.4.1

- There are different types of auctions we study. Their definitions are summarized here.

- **Definition 4.4.1**

The different types of auctions are:

- **English Auction.** *Bids are announced publicly, and the bids rise until only one bidder is left. That bidder wins the object at the highest bid.*
- **Sealed Bid, First Price.** *Bids are private and are made simultaneously. The highest sealed bid wins and the winner pays that bid.*
- **Sealed Bid, Second Price.** *Bids are private and made simultaneously. The high bid wins, but the winner pays the second highest bid. This is also called a Vickrey auction after the Nobel Prize winner who studied them.*

Definition 4.4.1 (cont'd)

- **Dutch Auction.** *The auctioneer (which could be a machine) publicly announces a high bid. Bidders may accept the bid or not. The announced prices are gradually lowered until someone accepts that price. The first bidder who accepts the announced price wins the object and pays that price.*

- **Private Value Auction.** *Bidders are certain of their own valuation of the object up for auction and these valuations (which may be random variables) are independent.*

- **Common Value Auction.** *The object for sale has the same value (that is not known for certain to the bidders) to all the bidders. Each bidder has their own estimate of this value.*

EXAMPLE 4.5

- In an online auction with no middleman the seller of the object and the buyer of the object may choose to renege on the deal dishonestly or go through with the deal honestly.
  - The buyer could choose to wait for the item and then not pay for it. The seller could simply receive payment but not send the item.
  - Here is a possible payoff matrix for the buyer and the seller:

Buyer/Seller	Send	Keep
Pay	$(1, 1)$	$(-2, 2)$
Don't Pay	$(2, -2)$	$(-1, -1)$

There is only one Nash equilibrium in this problem and it is at (don't pay,keep); neither player should be honest!

EXAMPLE 4.5 (cont'd)

- Amazing, the transaction will never happen and it is all due to either lack of trust on the part of the buyer and seller.

- Now let's introduce an auction house that serves two purposes:
  1. It guar-antees payment to the seller.
  2. It guarantees delivery of the item for the buyer.
  - This introduces a third strategy for the buyer and seller to use: Auctioneer. This changes the payoff matrix as follows:

Buyer/Seller	Send	Keep	Auctioneer
Pay	$(1,1)$	$(-2,2)$	$(1,1-c)$
Don't pay	$(2,-2)$	$(-1,-1)$	$(0,-c)$
Auctioneer	$(1-c,1)$	$(-c,0)$	$(1-c,1-c)$

EXAMPLE 4.5 (cont'd)

- The idea is that each player has the **option, but not the obligation,** of using an auctioneer.
- If somehow they should agree to both be honest, they both get +1. If they both use an auctioneer, the auctioneer will charge a fee of $0 < c < 1$ and the payoff to each player will be $1 - c$.

- We use a calculus procedure to find the mixed Nash equilibrium for this symmetric game as a function of c.
  - The result of this calculation is

$$X_c = \left(\frac{1}{2}(1-c), \frac{1}{2}c, \frac{1}{2}\right) = Y_c$$

and (X_c, Y_c) is the unique Nash equilibrium.

EXAMPLE 4.5 (cont'd)

- The expected payoffs to each player are

$$E_{\text{I}}(X_c, Y_c) = 1 - \frac{3}{2}c = E_{\text{II}}(X_c, Y_c).$$

- As long as $\frac{2}{3} > c > 0$, both players receive a positive expected payoff.

- The Nash equilibrium tells the buyer and seller to use the auctioneer half the time, no matter what value c is.
  - Each player should be dishonest with probability c/2, which will increase as c increases.
  - The players should be honest with probability only (1 - c)/2. If c = 1 they should never play honestly and either play dishonestly or use an auctioneer half the time.

EXAMPLE 4.5 (cont'd)

- You can see that the existence and only the existence of an auctioneer will permit the transaction to go through.
  - From an economics perspective, this implies that auctioneers **will** come into existence as an economic necessity for online auctions and it can be a very profitable business.

EXAMPLE 4.6

- A common feature of auctions is that the seller of the object may set a price, called the reserve price, so that if none of the bids for the object are above the **reserve price** the seller will not sell the object.

- Assume that the auction has two possible buyers. The seller must have some information, or estimates, about how the buyers value the object.
  - Suppose that the seller feels that each buyer values the object at either \$s (small amount) or \$*L* (large amount) with probability $\frac{1}{2}$ each.

EXAMPLE 4.6 (cont'd)

- Assuming bids may go up by a minimum of \$1, the winning bids with no reserve price set are \$s, \$(s + 1), \$(s + 1), or \$*L* each with probability $\frac{1}{4}$.
  - Without a reserve price, the expected payoff to the seller is

$$\frac{s + 2(s+1) + L}{4} = \frac{3s + 2 + L}{4}.$$

- Suppose next that the seller sets a reserve price at the higher valuation \$L, and this is the lowest acceptable price to the seller.
  - Let $B_i, i = 1, 2$ denote the random variable that is the bid for buyer *i.* Without collusion or passing of information we may assume that the B_i values are independent.

EXAMPLE 4.6 (cont'd)

- The seller is assuming that the valuations of the bidders are (s, s), (s, *L*),(*L,* s) and (*L*, *L*), each with probability $\frac{1}{4}$.
- If the reserve price is set at $*L*, the sale will not go through 25% of the time, but the expected payoff to the seller will be

$$\left(\frac{3}{4}\right) L + \left(\frac{1}{4}\right) 0 = \frac{3L}{4}.$$

- The question is whether it can happen that there are valuations s and *L,* so that

$$\frac{3L}{4} > \frac{3s + 2 + L}{4}.$$

  - Solving for *L,* the requirement is that *L* > (3s + 2)/2, and this certainly has solutions.

EXAMPLE 4.6 (cont'd)

– For example, if L = 100, any lower valuation s < 66 will lead to a higher expected payoff to the seller with a reserve price set at $ 100. Of course, this result depends on the valuations of the seller.

EXAMPLE 4.7

- If only one bidder shows up for your auction and you must sell to the high bidder, then you do not make any money at all unless you set a reserve price.
  - Assume that a buyer has a valuation of your gizmo at $*V*, where *V* is a random variable with cumulative distribution function *F*(*v*). Then, if your reserve price is set at $*p*, the expected payoff (assuming one bidder) will be

$$\begin{aligned} E[\text{Payoff}] &= pProb(V > p) + 0Prob(V \leq p) \\ &= pProb(V > p) = p(1 - F(p)). \end{aligned}$$

  - You want to choose p to maximize this function $g(p) \equiv p(1 - F(p))$.

EXAMPLE 4.7 (cont'd)

- Assuming that $f(p) = F'(p)$ is the probability density function of V:

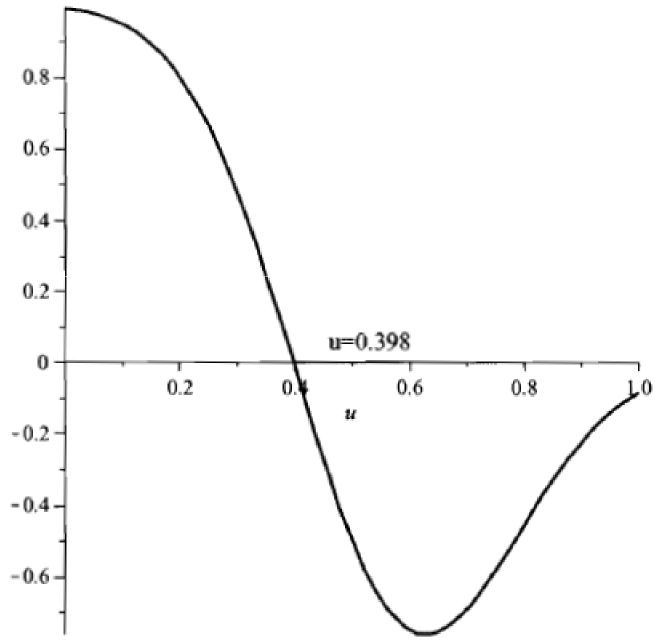
$$g'(p) = (1 - F(p)) - pf(p) = 0.$$

  Assuming that $g''(p) = 2f(p) + pf'(p) \geq 0$, we will have a maximum.
- We need to solve $1 - F(p) - pf(p) = 0$, and assume that the random variable *V* has a normal distribution with mean 0.5 and standard deviation 0.2.
- This is easy to do using Maple. The Maple commands to get our result are

```
> restart:
> with(Statistics):
> X := RandomVariable(Normal(0.5,0.2)):
> plot(1-CDF(X,u)-u*PDF(X,u),u=0..1);
> fsolve(1-CDF(X,u)-u*PDF(X,u),u);
```

EXAMPLE 4.7 (cont'd)

– The plot $b(u) = 1 - F(u) - p * f(u)$ is shown in the following figure.



We see that the function $b(u)$ crosses the axis at $u = p^* = 0.398$. The reserve price should be set at 39.8% of the maximum valuation.

4.4.1 Complete Information Problems

Complete Information Problems

- Let's get into the theory from the bidders' perspective.
  - There are *N* bidders (^players) in this game. There is one item up for bid and each player **values the object** at $v_1 \geq v_2 \geq \cdots \geq v_N > 0$ dollars.
  - One question we have to deal with is whether the bidders know this ranking of values.

- In the simplest and almost totally unrealistic model, all the bidders have complete information about the valuations of all the bidders.

Definition 4.4.2

- **Definition 4.4.2**

  A first-price, sealed-bid auction is an auction in which each bidder submits a bid $b_i, i = 1, 2, \ldots, N$ in a sealed envelope. After all the bids are received the envelopes are opened by the auctioneer and the person with the highest bid wins the object and pays the bid b_i. If there are identical bids, the winner is chosen at random from the identical bidders.

  – If bidder *i* doesn't win the object, she pays nothing and gets nothing. That will occur if she is not a high bidder:

  $$b_i < \max\{b_1, \ldots, b_N\} \equiv M.$$

  On the other hand, if she is a high bidder, so that

  $$b_i = M$$

Payoff

- The payoff is the difference between what she bid and what she thinks it's worth $v_i - b_i$.
  - If she bids less than her valuation of the object, and wins the object, then she gets a positive payoff, but she gets a negative payoff if she bids more than it's worth to her.
  - To take into account the case when there are *k* ties in the high bids, she would get the average payoff.
    - Let's use the notation that {*k*} is the set of high bidders. So, in symbols

$$u_i(b_1, \ldots, b_N) = \begin{cases} 0 & \text{if } b_i < M \text{, she is not a high bidder;} \\ v_i - b_i & \text{if } b_i = M \text{, she is the sole high bidder;} \\ \dfrac{v_i - b_i}{k} & \text{if } i \in \{k\} \text{, she is one of } k \text{ high bidders.} \end{cases}$$

The Nash Equilibrium for This Game

1. Each bidder should bid $b_i \leq v_i, i = 1, 2, \ldots, N$. Never bid more than the valuation.
   – To see this, just consider the following cases.

     If player *i* bids $b_i > v_i$ and wins the auction, then $u_i < 0$, even if there are ties. If player *i* bids $b_i > v_i$ and does not win the auction, then $u_i = 0$. But if player *i* bids $b_i \leq v_i$,*in* all cases $u_i \geq 0$.

2. In the case when the highest valuation is strictly bigger than the second highest valuation, $v_1 > v_2$ player 1 bids $b_1 \approx v_2, v_1 > b_1 > v_2$ that is, player 1 wins the object with any bid greater than v_2 and so should bid very close to but higher than v_2.

The Nash Equilibrium for This Game (cont'd)

– Notice that this is an open interval and the maximum is not actually achieved by a bid.
– If bidding is in whole dollars, then $b_1 = v_2 + 1$ is the optimal bid.
– There is no Nash equilibrium achieved in the case where the winning bid is in $(v_2, v_1]$ because it is an open interval at v_2.

3. In the case when $v_1 = v_2 = \cdots = v_k$, so there are *k* players with the highest, but equal, value of the object, then player *i* should bid v_i . So, the bid $(b_1, \ldots, b_N) = (v_1, \ldots, v_N)$ will be a Nash equilibrium inthis case.

4.4.2 Incomplete Information

Incomplete Information

- The problem is that the valuations are not known to either the buyers or the seller, except for their own valuation.
  - We assume that the seller and buyers think of the valuations as random variables.
  - The information that we assume known to the seller is the joint cumulative distribution function

$$F(v_1, v_2, \ldots, v_N) = Prob(V_1 \leq v_1, \ldots, V_n \leq v_N),$$

    and each buyer *i* has knowledge of his or her own distribution function.

Take-It-or-Leave-It Rule

- This is the simplest possible problem that may still be considered an auction. But it is not really a game. It is a problem of the seller of an object as to how to set the **buy-it-now** price.

- In an auction you may set a **reserve price** r, which, as we have seen, is a nonnegotiable lowest price you must get to consider selling the object. You may also declare a price $p \geq r$, which is your **take-it-or-leave-it price** or **buy-it-now price** and wait for some buyer, who hopefully has a valuation greater than or equal to *p* to buy the object. The problem is to determine *p*.

Take-It-or-Leave-It Rule (cont'd)

- The solution involves calculating the expected payoff from the trade and then maximizing the expected payoff over $p \geq r$. The payoff is the function $U(p)$, which is p - r, if there is a buyer with a valuation at least p, and 0 otherwise:

$$U(p) = \begin{cases} p - r & \text{if } \max\{V_1, \ldots, V_N\} \geq p; \\ 0 & \text{if } \max\{V_1, \ldots, V_N\} < p. \end{cases}$$

  - This is a random variable because it depends on $V_1, \ldots, V_N$, which are random.

Take-It-or-Leave-It Rule (cont'd)

- The expected payoff is

$$\begin{aligned} u(p) = E[U(p)] &= (p-r)P(\max\{V_1,\ldots,V_N\} \geq p) + 0 \cdot P(\max\{V_1,\ldots,V_N\} < p) \\ &= (p-r)[1 - P(\max\{V_1,\ldots,V_N\} < p)] \\ &= (p-r)[1 - P(V_1 < p, \cdots, V_N < p)] \\ &= (p-r)[1 - F(p,p,\ldots,p)] \\ &= (p-r)f(p), \;\; f(p) = 1 - F(p,p,\ldots,p) \end{aligned}$$

  - The seller wants to find p^* such that

$$\max_{p \geq r}(p-r)f(p) = (p^*-r)f(p^*).$$

Take-It-or-Leave-It Rule (cont'd)

- If there is a maximum $p^* > r$, we could find it by calculus. It is the solution of

$$(p^* - r)f'(p^*) + f(p^*) = 0.$$

  This will be a maximum as long as $(p^* - r)f''(p^*) + 2f'(p^*) \leq 0$.

- To proceed further, we need to know something about the valuation distribution.
  - Let's take the simplest case that $\{V_i\}$ is a collection of N independent and identically distributed random variables.

Take-It-or-Leave-It Rule (cont'd)

– In this case

$$F(v_1, \ldots, v_N) = G(v_1) \cdots G(v_N), \text{where } G(v) = F_i(v), 1 \leq i \leq N.$$

Then

$$f(p) = 1 - G(p)^N, \quad f'(p) = -NG(p)^{N-1}G'(p), \quad \text{and} \quad G'(p) = g(p)$$

is the density function of V_i, if it is a continuous random variable.

– So the condition for a maximum at p^* becomes

$$-(p^* - r)NG(p^*)^{N-1}g(p^*) + 1 - G(p^*)^N = 0.$$

Take-It-or-Leave-It Rule (cont'd)

– Now we take a particular distribution for the valuations that is still realistic. In the absence of any other information, we might as well assume that the valuations are uniformly distributed over the interval [*r*, *R*]. For this uniform distribution. We have

$$g(p) = \begin{cases} \dfrac{1}{R-r} & r < p < R; \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad G(p) = \begin{cases} 0 & \text{if } p < r; \\ \dfrac{p-r}{R-r} & \text{if } r < p < R; \\ 1 & \text{if } p > R. \end{cases}$$

– If we assume that $r < p^* < R$, then we may solve

$$-(p^* - r)N\left(\frac{p^*-r}{R-r}\right)^{N-1}\left(\frac{1}{R-r}\right) + 1 - \left(\frac{p^*-r}{R-r}\right)^{N} = 0$$

Take-It-or-Leave-It Rule (cont'd)

for *p** to get the take-it-or-leave-it price

$$p^* = r + (R - r)\left(\frac{1}{N+1}\right)^{1/N}.$$

For this *p** we have the expected payoff

$$u(p^*) = (p^* - r)f(p^*) = (R - r)N\left(\frac{1}{N+1}\right)^{N+1}$$

Take-It-or-Leave-It Rule (cont'd)

- Of particular interest are the cases *N* = 1, *N* = 2, and $N \to \infty$. We label the take-it-or-leave-it price as *p** - *p** (*N*). Here are the results:
  1. When there is only one potential buyer the take-it-or-leave-it price should be set at

     $$p^*(1) = r + \frac{R - r}{2} = \frac{r + R}{2}$$

     the midpoint of the range [*r*, *R*]. The expected payoff to the seller is

     $$u(p^*(1)) = \frac{R - r}{4}.$$

Take-It-or-Leave-It Rule (cont'd)

2. When there are two potential buyers, the take-it-or-leave-it price should be set at

$$p^*(2) = r + \frac{R-r}{\sqrt{3}}, \text{ and then } u(p^*(2)) = (R-r)\frac{2\sqrt{3}}{9}.$$

3. As $N \to \infty$, we have the take-it-or-leave-it price should be set at

$$p^*(\infty) = \lim_{N\to\infty} p^*(N) = \lim_{N\to\infty} r + (R-r)\left(\frac{1}{N+1}\right)^{1/N} = R,$$

   and then the expected payoff is $u(p^*(\infty)) = R - r$. Notice that we may calculate $\lim_{N\to\infty}(1/N+1)^{1/N} = 1$ using L'Hopital's rule. We conclude that as the number of potential buyers increases, the price should be set at the upper range of valuations.

4.4.3 Symmetric Independent Private Value Auctions Problems

Symmetric Independent Private Auctions Value Problems

- It is assumed that the unknown valuations $V_1,.,. ,V_N$ are independent and identically distributed continuous random variables.
  - The **symmetric** part of the title of this section comes from assuming that the bidders all have the same valuation distribution.

- We will consider two types of auction:
  - English auction, where bids increase until everyone except the highest one or two bidders are gone. In a first-price auction the high bidder gets the object.
  - Dutch auction, where the auctioneer asks a price and lowers the price continu- ously until one or more bidders decide to buy the item at the latest announced price. In a tie the winner is chosen randomly.

Dutch Auction & First-Price Sealed-Bid Auction

- A Dutch auction is equivalent to the first-price sealed-bid auction.
  - In both cases the bidder must decide the highest price she is willing to pay and submit that bid.
  - In a Dutch auction the object will be awarded to the highest bidder at a price equal to her bid. That is exactly what happens in a first-price sealed-bid auction.
  - The strategies for making a bid are identical in each of the two seemingly different types of auction.

English Auction & Second-price Sealed-bid Auction

- An English auction can also be shown to be equivalent to a **second-price sealed-bid auction** (as long as we are in the private values set up).
  - As long as bidders do not change their valuations based on the other bidders' bids, which is assumed, then bidders should accept to pay any price up to their own valuations.
  - A player will continue to bid until the current announced price is greater than how much the bidder is willing to pay. This means that the item will be won by the bidder who has the highest valuation and she will win the object at a price equal to the **second highest valuation.**

English Auction & Second-price Sealed-bid Auction (cont'd)

– A bidder should submit a bid that is equal to her valuation of the object since she is willing to pay an amount less than her valuation but not willing to pay a price greater than her valuation. Assuming that each player does that, the result of the auction is that the object will be won by the bidder with the highest valuation and the price paid will be the second highest valuation.

Symmetric Independent Private Value Auctions

- For simplicity it will be assumed that the reserve price is normalized to r = 0. As above, we assume that the joint distribution function of the valuations is given by

$$F_J(v_1, \ldots, v_N) = Prob(V_1 \leq v_1, \ldots, V_N \leq v_N) = F(v_1)F(v_2)\cdots F(v_N),$$

  which holds because we assume independence and identical distribution of the bid-der's valuations.

Symmetric Independent Private Value Auctions

(cont'd)

- Suppose that the maximum possible valuation of all the players is a fixed constant $w > 0$. Then, a bidder gets to choose a bidding function $b_i(v)$ that takes points in $[0, w]$ and gives a positive real bid.
  - Because of the symmetry property, it should be the case that all players have the same payoff function and that all optimal strategies for each player are the same for all other players.

- Now suppose that we have a player with bidding function $b = b(v)$ for a valuation $v \in [0, w]$. The payoff to the player is given by

$$u(b(v), v) = Prob(b(v) \text{ is high bid})v - E(\text{payment for bid } b(v)).$$

  We have to use the expected payment for the bid $b(v)$ because we don't know whether the bid $b(v)$ will be a winning bid.

Symmetric Independent Private Value Auctions
(cont'd)

– We will simplify notation a bit by setting

$$f(b(v)) = \ Prob(b(v) \text{ is high bid}) = Prob(b(v) > \max\{b(V_1), \ldots, b(V_N)\})$$

and then rewrite the payoff as

$$u(b(v), v) = f(b(v))v - E(b(v)).$$

– We want the bidder to maximize this payoff by choosing a bidding strategy $\beta(v)$.
  - One property of $\beta(v)$ should be obvious, namely, as the valuation increases, the bid must increase. The fact that $\beta(v)$ is strictly increasing as a function of *v.*

Symmetric Independent Private Value Auctions

(cont'd)

- Let's take the specific example that bidder's valuations are uniform on $[0, w] = [0, 1]$. Then for each player, we have

$$F(v) = \begin{cases} 0 & \text{if } v < 0; \\ v & \text{if } 0 \leq v \leq 1; \\ 1 & \text{if } v > 1. \end{cases}$$

Remark. Whenever we say in the following that we are normalizing to the interval $[0, 1]$, this is not a restriction because we may always transform from an interval $[r, R]$ to $[0, 1]$ and the reverse by the linear transformation $t = (s - r)/(R - r),\ r \leq s \leq R$, or $s = r + (R - r)t, 0 \leq t \leq 1$.

Theorem 4.4.3

- **Theorem 4.4.3**

Suppose that valuations $V_1, \ldots, V_N$, are uniformly distributed on $[0,1]$, and the expected payoff function for each bidder is

$$u(b(v), v) = f(b(v))v - E(b(v)), \text{ where} \tag{4.4.1}$$
$$f(b(v)) = Prob(b(v) > \max\{b(V_1), \ldots, b(V_N)\}).$$

Then there is a unique Nash equilibrium $(\beta, \ldots, \beta)$ given by

$$\beta(v_i) = v_i, \quad i = 1, 2, \ldots, N$$

in the case when we have an English auction, and

$$\beta(v_i) = \left(1 - \frac{1}{N}\right) v_i, \quad i = 1, 2, \ldots, N,$$

in the case when we have a Dutch auction. In either case the expected payment price for the object is

$$p^* = \frac{N-1}{N+1}.$$

Theorem 4.4.3 (cont'd)

Proof. To see why this is true, we start with the Dutch auction result. Since all players are indistinguishable, we might as well say our that guy is player 1. Suppose that player 1 bids $b = \beta(v)$. Then the probability that she wins the object is given by

$$Prob(\beta(\max\{V_2, \dots, V_N\}) < b).$$

Now here is where we use the fact that β is strictly increasing, because then it has an inverse, $v = \beta^{-1}(b)$, and so we can say

$$\begin{aligned} f(b) &= Prob(\beta(\max\{V_2, \dots, V_N\}) < b) \\ &= Prob(\max\{V_2, \dots, V_N\} < \beta^{-1}(b)) = Prob(V_i < \beta^{-1}(b), i = 2, \dots, N) \\ &= F(\beta^{-1}(b))^{N-1} = [\beta^{-1}(b)]^{N-1} = v^{N-1}, \end{aligned}$$

because all valuations are independent and identically distributed. The next-to-last-equality is because we are assuming a uniform distribution here. The function $f(b)$ is the probability of winning the object with a bid of b.

Theorem 4.4.3 (cont'd)

Now, for the given bid $b = \beta(v)$, player 1 wants to maximize her expected payoff. The expected payoff (4.4.1) becomes

$$\begin{aligned} u(\beta(v), v) &= f(\beta(v))v - E(\beta(v)) \\ &= f(b)v - (bProb(\text{win}) + 0 \cdot Prob(\text{lose})) \\ &= f(b)v - bf(b) = (v - b)f(b). \end{aligned}$$

Taking a derivative of $u(b, v)$ with respect to b, evaluating at $b = \beta(v)$ and setting to zero, we get the condition

$$f'(\beta)(v - \beta) - f(\beta) = 0. \tag{4.4.2}$$

Since $f(b) = [\beta^{-1}(b)]^{N-1} = v^{N-1}, v = \beta^{-1}(b)$, we have

$$\frac{df(b)}{db} = (N-1)[\beta^{-1}(b)]^{N-2} \frac{d\beta^{-1}(b)}{db}.$$

Theorem 4.4.3 (cont'd)

Therefore, after dividing out the term $[\beta^{-1}(b)]^{N-2}$, the condition (4.4.2) becomes

$$(N-1)[\beta^{-1}(b)-b]\frac{d\beta^{-1}(b)}{db}-\beta^{-1}(b)=0.$$

Let's set $y(b)=\beta^{-1}(b)$ to see that this equation becomes

$$(N-1)[y(b)-b]y'(b)-y(b)=0. \qquad (4.4.3)$$

This is a **first-order ordinary differential equation** for $y(b)$, that we may solve to get

$$y(b)=\beta^{-1}(b)=v=\frac{N}{N-1}b.$$

Theorem 4.4.3 (cont'd)

Actually, we may find the general solution of (4.4.3) using Maple fairly easily. The Maple commands to do this are

```
> ode:=a*(y(x)-x)*diff(y(x),x)=y(x);
> dsolve(ode,y(x));
```

That's it, just two commands without any packages to load. We are setting $a = N-1$. Here is what Maple gives you:

$$x - \frac{a}{a+1}y(x) - (y(x))^{-a}\,\_C1 = 0.$$

The problem with this is that we are looking for the solution with $y(0) = 0$ because when the valuation of the item is zero, the bid must be zero. Notice that the term $y(x)^{-a}$ is undefined if $y(0) = 0$. Consequently, we must take $C1 = 0$, leaving us with $x - (a/(a+1))y(x) = 0$ which gives $y(x) = ((a+1)/a)x$. Now we go back to our original variables to get $y(b) = v = \beta^{-1}(b) = (N/(N-1))b$. Solving for b in terms of v we get

$$b = \beta(v) = \left(1 - \frac{1}{N}\right)v,$$

which is the claimed optimal bidding function in a Dutch auction.

Theorem 4.4.3 (cont'd)

Next we calculate the expected payment. In a Dutch auction, we know that the payment will be the highest bid. We know that is going to be the random variable $\beta(\max\{V_1, \ldots, V_N\})$, which is the optimal bidding function evaluated at the largest of the random valuations. Then

$$\begin{aligned} E(\beta(\max\{V_1, \ldots, V_N\})) &= E\left[\left(1 - \frac{1}{N}\right)\max\{V_1, \ldots, V_N\}\right] \\ &= \left(1 - \frac{1}{N}\right)E[\max\{V_1, \ldots, V_N\}] \\ &= \frac{N-1}{N+1} \end{aligned}$$

because $E[\max\{V_1, \ldots, V_N\}] = N/(N+1)$, as we will see next, when the V_i values are uniform on $[0, 1]$.

Theorem 4.4.3 (cont'd)

Here is why $E[\max\{V_1, \ldots, V_N\}] = N/(N+1)$. The cumulative distribution function of $Y = \max\{V_1, \ldots, V_N\}$ is derived as follows. Since the valuations are independent and all have the same distribution,

$$F_Y(x) = Prob(\max\{V_1, \ldots, V_N\} \leq x) = P(V_i \leq x)^N = F_V(x)^N.$$

Then the density of Y is

$$f_Y(x) = F_Y'(x) = N(F_V(x))^{N-1} f_V(x).$$

In case V has a uniform distribution on $[0,1]$, $f_V(x) = 1, F_V(x) = x, 0 < x < 1$, and so

$$f_Y(x) = Nx^{N-1}, 0 < x < 1 \implies E[Y] = \int_0^1 x f_Y(x)\, dx = \frac{N}{N+1}.$$

- So we have verified everything for the Dutch auction case, but not for English auction yet.

Theorem 4.4.4

- Next we solve the English auction game. We have already discussed informally that in an English auction each bidder should bid his or her true valuation so that $\beta(v) = v$. Here is a formal statement and proof.

- **Theorem 4.4.4**

  In an English auction with valuations $V_1, \ldots, V_N$ and $V_i = v_i$ known to player i, then player i's optimal bid is v_i.

Theorem 4.4.4 (cont'd)

Proof.

1. Suppose that player i bids $b_i > v_i$, more than her valuation. Recall that an English auction is equivalent to a second-price sealed-bid auction. Using that we calculate her payoff as

$$u_i(b_1, \ldots, b_i, \ldots, b_N) \tag{4.4.4}$$
$$= \begin{cases} v_i - \max_{k \neq i} b_k \geq 0 & \text{if } v_i \geq \max_{k \neq i} b_k; \\ v_i - \max_{k \neq i} b_k < 0 & \text{if } v_i < \max_{k \neq i} b_k < b_i; \\ 0 & \text{if } \max_{k \neq i} b_k > b_i. \end{cases}$$

(We are ignoring possible ties.) To see why this is her payoff, if $v_i \geq \max_{k \neq i} b_k$, then player i's valuation is more than the bids of all the other players, so she wins the auction (since $b_i > v_i$) and pays the highest bid of the other players (which is $\max_{k \neq i} b_k$), giving her a payoff of $v_i - \max_{k \neq i} b_k > 0$.

If $v_i < \max_{k \neq i} b_k < b_i$, she values the object as less than at least one other player but bids more than this other player. So she pays $\max_{k \neq i} b_k$, and her payoff is $v_i - \max_{k \neq i} b_k < 0$. In the last case, she does not win the object, and her payoff is zero.

Theorem 4.4.4 (cont'd)

2. If player i's bid is $b_i \le v_i$, then her payoff is

$$u_i(b_1, \ldots, b_i, \ldots, b_N) \tag{4.4.5}$$
$$= \begin{cases} v_i - \max_{k \neq i} b_k > 0 & \text{if } v_i \ge b_i > \max_{k \neq i} b_k; \\ 0 & \text{if } \max_{k \neq i} b_k > b_i. \end{cases}$$

So, in all cases the payoff function in (4.4.5) is at least as good as the payoff in (4.4.4) and better in some cases. Therefore, player i should bid $b_i \le v_i$.

3. Now what happens if player i bids $b_i < v_i$? In that case,

$$u_i(b_1, \ldots, b_i, \ldots, b_N)$$
$$= \begin{cases} v_i - \max_{k \neq i} b_k > 0 & \text{if } v_i > b_i > \max_{k \neq i} b_k; \\ 0 & \text{if } \max_{k \neq i} b_k > b_i. \end{cases}$$

Player i will get a strictly positive payoff if $v_i > b_i > \max\{b_k : k \neq i\}$.

Theorem 4.4.4 (cont'd)

Putting steps (3) and (2) together, we therefore want to maximize the payoff for player i using bids b_i subject to the constraint $v_i \geq b_i > \max_{k \neq i} b_k$. Looking at the payoffs, the biggest that u_i can get is when $b_i = v_i$. In other words, since $b_i < v_i$ and the probability increases,

$$Prob(b_i > \max_{k \neq i} b_k(V_k)) \leq Prob(v_i > \max_{k \neq i} b_k(V_k)).$$

The largest probability, and therefore the largest payoff, will occur when $b_i = v_i$. $\square$

Theorem 4.4.3 (cont'd)

- Having proved that the optimal bid in an English auction is $b_i = \beta(v_i) = v_i$, we next calculate the expected payment and complete the proof of Theorem 4.4.3.

Proof.

The winner of the English auction with uniform valuations makes the payment of the second highest bid, which is given by

$$\text{If } V_j = \max\{V_1, \ldots, V_N\}, \text{ then } E[\max_{i \neq j}\{V_i\}] = \frac{N-1}{N+1}.$$

This follows from knowing the density of the random variable $Y = \max_{i \neq j}\{V_i\}$, the second highest valuation. In the case when V is uniform on $[0, 1]$ the density of Y is

$$f_Y(x) = N(N-1)x(1-x)^{N-1}, 0 < x < 1 \implies E[Y] = \frac{N-1}{N+1}.$$

You can refer to the Appendix B for a derivation of this using order statistics. Therefore we have proved all parts of Theorem 4.4.3. □

Risk of English and Dutch auctions

- One major difference between English and Dutch auctions is the risk character-istics as measured by the variance of the selling price.
  1. In an English auction, the selling price random variable is the second highest valuation, that we write as $P_E = \max_2\{V_1, \ldots, V_N\}$.

     In probability theory this is an **order statistic** (see Appendix B), and it is shown that if the valuations are all uniformly distributed on [0,1], then

$$Var(P_E) = \frac{2(N-1)}{(N+1)^2(N+2)}.$$

Risk of English and Dutch auctions (cont'd)

2. In a Dutch auction, equivalent to a first-price sealed-bid auction, the selling price is $P_D = \beta(\max\{V_1, \ldots, V_N\})$, and we have seen that with uniform valuations

$$\beta(\max\{V_1, \ldots, V_N\}) = \frac{N-1}{N} \max\{V_1, \ldots, V_N\}.$$

Consequently

$$\begin{aligned} Var(P_D) &= Var(\beta(\max\{V_1, \ldots, V_N\})) \\ &= \left(\frac{N-1}{N}\right)^2 Var(\max\{V_1, \ldots, V_N\}) \\ &= \frac{(N-1)^2}{N(N+1)^2(N+2)}. \end{aligned}$$

Risk of English and Dutch auctions (cont'd)

- We claim that $Var(P_D) < Var(P_E)$. That will be true if

$$\frac{2(N-1)}{(N+1)^2(N+2)} > \frac{(N-1)^2}{N(N+1)^2(N+2)}.$$

  - After using some algebra, this inequality reduces to the condition $2 > \frac{(N-1)}{N}$, which is absolutely true for any $N \geq 1$.
  - We conclude that Dutch auctions are less risky for the seller than are English auctions, as measured by the variance of the payment.

The Valuations Are Not Uniformly Distributed

- The problem will be much harder to solve explicitly.
  - But there is a general formula for the Nash equilibrium still assuming independence and that each valuation has distribution function *F*(*v*).
    - If the distribution is continuous, the Dutch auction will have a unique Nash equilibrium given by

$$\beta(v) = v - \frac{1}{F(v)^{N-1}} \int_r^v F(y)^{N-1}\, dy.$$

The proof of this formula comes basically from having to solve the differential equation that we derived earlier for the Nash equilibrium

$$(N-1)f(y(b))(y(b)-b)y'(b) - F(y(b)) = 0,$$

$y(b) = \beta^{-1}(b)$, $F(y)$: cdf , $f = F'$: pdf

Theorem 4.4.5

- The expected payment in a Dutch auction and English auction with uniformly distributed valuations were shown to be

$$E[P_D] = (N-1)/(N+1) = E[P_E] = (N-1)/(N+1)$$

Is the expected payment for an auction always $(N-1)/(N+1)$, at least for valuations that are uniformly distributed? The answer is "Yes," and not just for uniform distributions.

Theorem 4.4.5 (cont'd)

- **Theorem 4.4.5**

  Any symmetric private value auction with identically distributed valuations, satisfying the following conditions, always has the same expected payment to the seller of the object:

  1. *They have the same number of bidders (who are risk-neutral).*
  2. *The object at auction always goes to the bidder with the highest valuation.*
  3. *The bidder with the lowest valuation has a zero expected payoff.*

  – This is known as the **revenue equivalence theorem.**

Linear Trading Rules

- **Linear trading rules** are the way to bid when the valuations are uniformly distributed in the interval [*r*, *R*], where *r* is the reserve price.
  - For simplicity we consider only two bidders who will have payoff functions

$$u_1((b_1, v_1), (b_2, v_2)) = \begin{cases} v_1 - b_1 & \text{if } b_1 > b_2; \\ \dfrac{v_1 - b_1}{2} & \text{if } b_1 = b_2; \\ 0 & \text{if } b_1 < b_2. \end{cases}$$

and

$$u_2((b_1, v_1), (b_2, v_2)) = \begin{cases} v_2 - b_2 & \text{if } b_2 > b_1; \\ \dfrac{v_2 - b_2}{2} & \text{if } b_1 = b_2; \\ 0 & \text{if } b_2 < b_1. \end{cases}$$

Linear Trading Rules (cont'd)

We have explicitly indicated that each player has two variables to work with, namely, the bid and the valuation.

– The independent valuations of each player are random variables V_1, V_2 with identical cumulative distribution function $F_V(v)$.

Each bidder knows his or her own valuation but not the opponent's. So the expected payoff to player 1 is

$$U_1(b_1, b_2) \equiv Eu_1(b_1, v_1, b_2(V_2), V_2) = Prob(b_1 > b_2(V_2))(v_1 - b_1)$$

- Because in all other cases the expected value is zero. In the case $b_1 = b_2(V_2)$ it is zero since we have continuous random variables.

Linear Trading Rules (cont'd)

– Similarly

$$U_2(b_1, b_2) \equiv Eu_2(b_1(V_1), V_1, b_2, v_2) = Prob(b_2 > b_1(V_1))(v_2 - b_2).$$

– A Nash equilibrium must satisfy

$$U_1(b_1^*, b_2^*) \geq U_1(b_1, b_2^*) \text{ and } U_2(b_1^*, b_2^*) \geq U_1(b_1^*, b_2).$$

– In the case that the valuations are uniform on [*r*, *R*], we will verify that the bidding rules

$$\beta_1^*(v_1) = \frac{r + v_1}{2} \text{ and } \beta_2^*(v_2) = \frac{r + v_2}{2}$$

constitute a Nash equilibrium.

Linear Trading Rules (cont'd)

Proof.

So, by the assumptions, we have

$$\begin{aligned} Prob(b_1 > \beta_2^*(V_2)) &= Prob\left(b_1 > \frac{r+V_2}{2}\right) \\ &= Prob(2b_1 - r > V_2) \\ &= \frac{(2b_1 - r) - r}{R - r}, \end{aligned}$$

if $r/2 < b_1 < (r+R)/2$, and the expected payoff

$$\begin{aligned} U_1(b_1, \beta_2^*(V_2)) &= (v_1 - b_1)Prob(b_1 > \beta_2^*(V_2)) \\ &= (v_1 - b_1)Prob(V_2 < 2b_1 - r) \\ &= \begin{cases} 0 & \text{if } b_1 < \frac{r}{2}; \\ (v_1 - b_1)\dfrac{2b_1 - 2r}{R - r} & \text{if } \frac{r}{2} < b_1 < \frac{(r+R)}{2}; \\ v_1 - b_1 & \text{if } \frac{(r+R)}{2} < b_1. \end{cases} \end{aligned}$$

Linear Trading Rules (cont'd)

We want to maximize this as a function of b_1. To do so, let's consider the case $r/2 < b_1 < (r + R)/2$ and set

$$g(b_1) = (v_1 - b_1)\frac{2b_1 - 2r}{R - r}.$$

The function g is strictly concave down as a function of b_1 and has a unique maximum at $\beta_1 = r/2 + v_1/2$, as the reader can readily verify by calculus. We conclude that $\beta_1^*(v_1) = (r + v_1)/2$ maximizes $U_1(b_1, \beta_2^*)$. This shows that $\beta_1^*(v_1)$ is a best response to $\beta_2^*(v_2)$. We have verified the claim.

□

Linear Trading Rules (cont'd)

Remark. A similar argument will work if there are more than two bidders. Here are the details for three bidders. Start with the payoff functions (we give only the one for player 1 since the others are similar):

$$u_1(b_1, b_2, b_3) = \begin{cases} v_1 - b_1 & \text{if } b_1 > \max\{b_2, b_3\}; \\ \dfrac{v_1 - b_1}{k} & \text{if there are } k = 2, 3 \text{ high bids;} \\ 0 & \text{otherwise.} \end{cases}$$

The expected payoff if player 1 knows her own bid and valuation, but V_2, V_3 are random variables is

$$\begin{aligned} U_1(b_1, b_2, b_3) &= Eu_1(b_1, b_2(V_2), b_3(V_3)) \\ &= Prob(b_1 > \max\{b_2(V_2), b_3(V_3)\})(v_1 - b_1). \end{aligned}$$

Assuming independence of valuations, we obtain

$$U_1(b_1, b_2, b_3) = Prob(b_1 > b_2(V_2))Prob(b_1 > b_3(V_3))(v_1 - b_1).$$

EXAMPLE 4.8

- Two players are bidding in a first-price sealed-bid auction for a 1901s United States penny, a very valuable coin for collectors.
  - Each player values it at somewhere between \$750K and \$1000K dollars with a uniform distribution (so $r = 750$, $R = 1000$).
  - In this case, each player should bid $\beta_i(v_i) = \frac{1}{2}(750 + v_i)$.
  - If player 1 values the penny at \$800K, she should optimally bid 775K. If bidder 2 has a higher valuation, say, at \$850K, then player 2 will bid 800K and win the penny.

EXAMPLE 4.8 (cont'd)

- On the other hand, if this were a second-price sealed-bid auction, equivalent to an English auction.
  - Each bidder would bid their own valuations.
  - In this case b_1 = \$800K and b_2 = \$850K. Bidder 2 still gets the penny, but the selling price is the same.
  - On the other hand, if player 1 valued the penny at \$775K, then b_1 = \$775K, and that would be the selling price. It would sell for \$25K less than in a first-price auction.

EXAMPLE 4.9

- **Application of Auctions to Bertrand's Model of Economic Competition**
  - If we look at the payoff for a bidder in a Dutch auction we have seen in Theorem 4.4.3

    $$u(b(v), v) = f(b(v))v - E(b(v)) = f(b)v - bf(b) = (v - b)f(b),$$

    where

    $$f(b(v)) = Prob(b(v) > \max\{b(V_1), \ldots, b(V_N)\}).$$

  - We consider Bertrand's model of competition between $N \geq 2$ identical firms. Assume that each firm has a constant and identical unit cost of production $c > 0$, and that each firm knows its own cost but not those of the other firms. For simplicity, we assume that each firm considers the costs of the other firms as random variables with a uniform distribution on $[0, 1]$.

EXAMPLE 4.9 (cont'd)

- A basic assumption in the Bertrand model of competition is that the firm offering the **lowest price** at which the gadgets are sold will win the market.
  - With that assumption, we may make the Bertrand model to be a Dutch auction with a little twist, namely, the buyers are bidding to win the **market** at the lowest selling price of the gadgets. So, if we think of the **market as the object up for auction**, each firm wants to bid for the object and obtain it at the **lowest possible selling price**, namely, the lowest possible price for the gadgets.
- The Bertrand model is really an inverse Dutch auction, so if we invert variables, we should be able to apply the Dutch auction results.

EXAMPLE 4.9 (cont'd)

- Call a valuation $v = 1 - c$.
- Then call the price to be set for a gadget as one minus the bid, $p = 1 - b$, or $b = 1 - p$.
- The expected payoff function in the Dutch auction is then

$$
\begin{aligned}
u(b(v), v) &= (v - b)f(b) \\
&= (1 - c - ((1 - p))f(1 - p) \\
&= (p - c)f(1 - p),
\end{aligned}
$$

where

$$
\begin{aligned}
f(1 - p) &= Prob(1 - p(c) > \max\{1 - p(C_1), \ldots, 1 - p(C_N)\}) \\
&= Prob(p(c) < \min\{p(C_1), \ldots, p(C_N)\}).
\end{aligned}
$$

EXAMPLE 4.9 (cont'd)

- Putting them together, we have

$$w(p, c) = (p - c)Prob(p(c) < \min\{p(C_1), \ldots, p(C_N)\})$$

  as the profit function to each firm, because $w(p, c)$ is the expected net revenue to the firm if the firm sets price *p* for a gadget and has cost of production c per gadget.

- Now in a Dutch auction we know that the optimal bidding function is $b(v) = ((N - 1)/N)v$, so that the optimal expected price to set for gadgets as a function of the production cost.

$$p^*(c) = 1 - b(c) = 1 - \frac{N-1}{N}v = 1 - \frac{N-1}{N}(1 - c) = \frac{c(N-1)+1}{N}.$$

EXAMPLE 4.9 (cont'd)

– The expected payment in a Dutch auction is

$$E(b(\max\{V_1, \ldots, V_N\})) = \frac{N-1}{N+1}.$$

So

$$\begin{aligned} E[p^*(C_1, \ldots, C_N)] &= 1 - E[b(\max\{V_1, \ldots, V_N\})] \\ &= 1 - \frac{N-1}{N+1} \\ &= \frac{2}{N+1}. \end{aligned}$$